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A PROTOCOL AND PROCEDURE FOR ASSESSING MULTIATTRIBUTE PREFERENCE FUNCTIONS

STANFORD UNIVERSITY

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ADVANCED DECISION TECHNOLOGY PROGRAM

CYBERNETICS TECHNOLOGY OFFICE
DEFENSE ADVANCED RESEARCH PROJECTS AGENCY
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RESEARCH REPORT No. EES DA-77-2

A PROTOCOL AND PROCEDURE FOR ASSESSING MULTIATTRIBUTE PREFERENCE FUNCTIONS

by

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DECISION ANALYSIS PROGRAM

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Principal Investigator

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SUMMARY

An important decision may hinge on the decision maker's preferences over the possible outcomes. In many problems the possible outcomes will appear to have several important dimensions. This dissertation provides some fundamental tools and methods for interpreting and assessing preferences over multiattribute outcomes. There are three steps in the development.

First, six preference measures are presented as tools for understanding and analyzing multiattribute preference functions. Each preference measure has a simple interpretation, describes a fundamental aspect of local preference attitude, and is well-defined for any smooth preference function.

Second, a hierarchy of independence conditions and delta properties is identified to provide an easy method for limiting the functional expression of a preference attitude. Each independence condition and delta property is easy to interpret and apply during assessment. The weakest independence condition that we consider, deterministic additive independence, requires that an additive form exist for ordinal preferences.

The six preference measures along with the independence conditions and delta properties stand together as a powerful technology for interpreting any smooth multiattribute preference function.

In the third step, the preference measures, independence conditions, and delta properties are assembled into a practical assessment procedure for multiattribute preference functions. This procedure is concisely summarized in a flow chart; it applies for any preference attitude that satisfies deterministic additive independence. A multiattribute risk

preference function is assessed for an actual decision maker as an illustration.

"In the beginner's mind there are many possibilities,

but in the expert's there are few."

--Shunryu Suzuki Roshi
Zen Buddhist

CONTENTS

CONTENTS (Cont)

FIGURES

FIGURES (Cont)

TABLES

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Chapter 1

INTRODUCTION

1.1 Introduction to Multiattribute Preferences

Decision analysis is a normative technology for structuring an important decision and choosing the best course of action. In practice, understanding the decision maker's preferences over the possible outcomes is vital. Outcomes that have more than one dimension are called multiattribute outcomes.

This dissertation establishes a protocol and a procedure for assessing a decision maker's preferences over multiattribute outcomes. By protocol we mean the formalities of the assessment process; these include the preference assessment flow chart in Chapter 4 and the theory supporting it in Chapters 2 and 3. By procedure we mean the actual series of assessment questions and the method of using them for encoding the decision maker's preference function, including its numerical parameters. The procedure is illustrated by an example in Chapter 4, in which we assess a multiattribute preference function for an actual decision maker.

There are two important classes of multiattribute preference functions: a value function expresses an ordinal preference attitude and is useful for evaluating deterministic outcomes; a utility function is a value function that contains enough information to evaluate uncertain outcomes as well.

The schematic iceberg in Figure 1.1 represents our current state of knowledge in multiattribute preferences. The methods in this dissertation are useful for interpreting and assessing any smooth utility

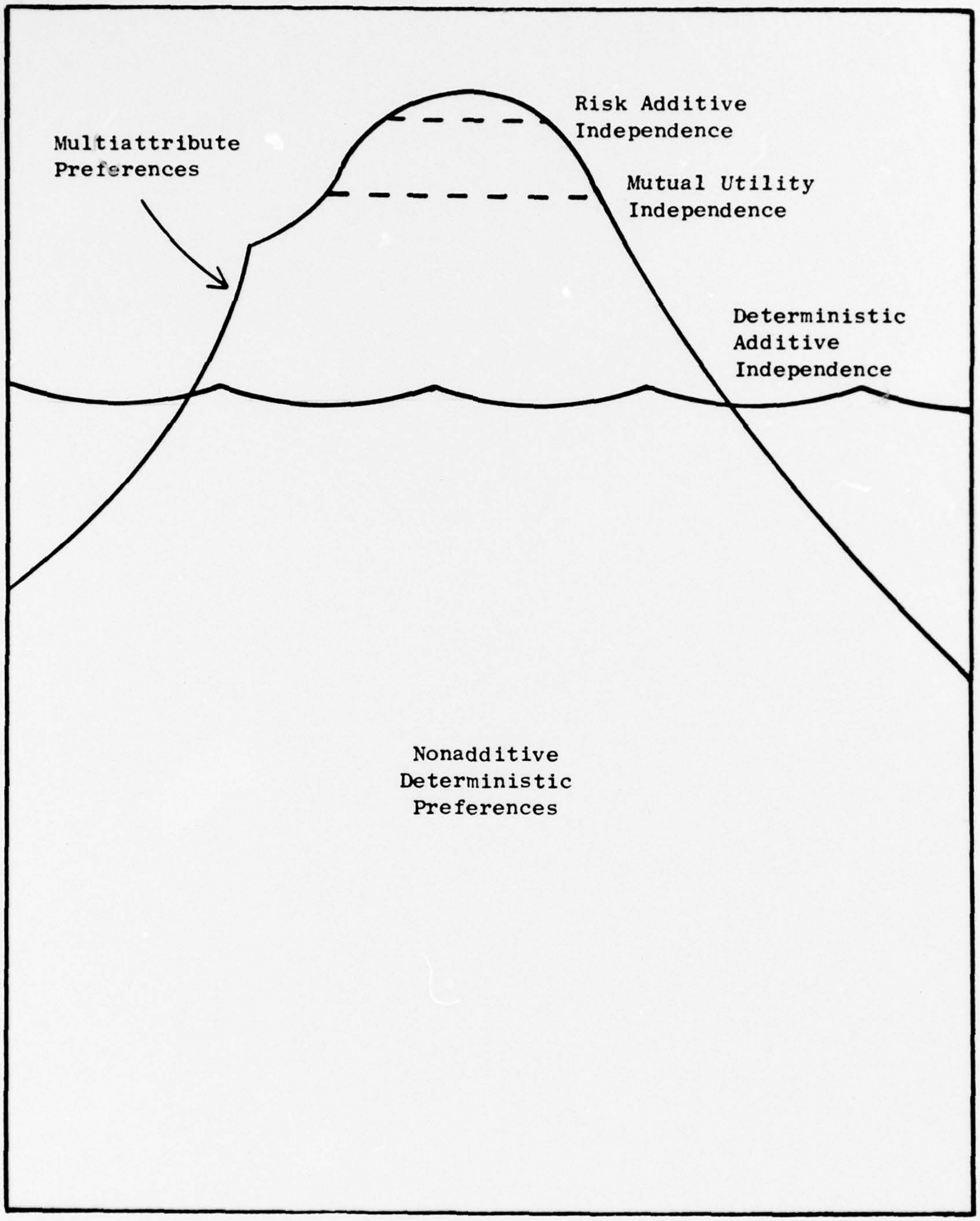


Figure 1.1. OUR CURRENT STATE OF KNOWLEDGE IN MULTIATTRIBUTE PREFERENCES.

function that lies above the water line. The independence conditions shown are discussed extensively in the literature and are used here to build a framework for preference assessment.

Below the water line, where ordinal preferences cannot be expressed in an additive form, our local preference measures for interpreting any smooth preference function still apply. However, we have yet to develop good methods for assessing nonadditive ordinal preferences.

1.2 Summary of Results and Contributions

Chapter 2 presents local interpretations for multiattribute preference functions, and Chapter 3 presents global interpretations. These chapters lay the theoretical foundation for the practical assessment technology in Chapter 4. The limitations of this technology suggest specific areas for research in Chapter 5.

Six local measures of multiattribute preference attitude are presented in Chapter 2. Each is easy to interpret and is well-defined wherever the preference function is twice differentiable. Three of the six measures apply to ordinal preferences and are unique under any increasing transformation of the value function; the other three are measures of risk preference and are unique under any positive linear transformation of the utility function.

An indifference curve is a locus of points in two dimensions among which a decision maker is indifferent. The marginal rate of substitution, the first ordinal preference measure in Section 2.2, is a well-known first-order approximation to an indifference curve. At each point it is defined as the negative slope of the tangent to the curve.

The next two definitions are my own and are useful for measuring local changes in the marginal rate of substitution.

The marginal value reduction coefficient is a measure of the percent reduction in the marginal rate of substitution when the decision maker becomes wealthier in one attribute while levels of the other attributes remain fixed. We show in Chapter 3 that the marginal value reduction coefficient does not depend on the fixed levels of the other attributes if and only if the value function can be written in an additive form. It is sometimes easier to interpret the reciprocal of the marginal value reduction coefficient, and we define this quantity as the marginal value preservation tolerance.

The substitution aversion coefficient is a second-order measure of the convexity of an indifference curve: at each point it measures the percent reduction in the marginal rate of substitution when moving along an indifference curve through that point. This coefficient has a natural graphical interpretation as the decision maker's local aversion to substitution at the marginal rate. The reciprocal of the substitution aversion coefficient is defined as the substitution tolerance.

Proceeding from the point of view that a multiattribute utility function can always be viewed as a utility function over a one-dimensional numeraire, we discuss three measures of risk aversion in Section 2.3. The first two are standard risk aversion coefficients as defined by Pratt (1964). The third is a new measure of multiattribute risk aversion.

The numeraire risk aversion coefficient simply measures risk attitude toward lotteries over the numeraire.

The single-attribute risk aversion coefficient is a measure of risk attitude toward lotteries over one attribute with the other attributes fixed.

The multiattribute risk aversion coefficient is a multidimensional analog to Pratt's risk aversion coefficient. We derive it using a Taylor approximation for a lottery's numeraire certain equivalent, and it provides us with a measure of multiattribute risk aversion, an aspect of risk aversion that does not exist in one dimension.

Having defined six preference measures, we discuss identities relating them in Section 2.4. Section 2.5 shows how to derive these measures most easily for a particular preference function, and Section 2.6 applies the preference measures to analyze an appealing method for constructing a risk preference function over cash flows.

Chapter 3 presents independence conditions and delta properties that are simple global interpretations of a multiattribute preference attitude. These are useful for limiting functional form during assessment.

Risk additive independence requires that a lottery can be evaluated entirely on the basis of its marginal distributions and implies that the utility function has an additive form. We show that risk additive independence is equivalent to global multiattribute risk indifference.

Mutual utility independence requires that the certain equivalent of a lottery over any subset of attributes does not depend on the fixed level of the remaining attributes and implies that the utility function must have either an additive or a multiplicative form.

A delta property is a simple lottery condition for limiting a utility function to a specific form. For additive utility functions, we can

look for a delta property in each individual term that would limit it to a linear, exponential, logarithmic, or power form. For nonadditive utility functions, we derive delta properties that limit functional form to a product-of-exponentials or a product-of-powers.

Deterministic additive independence implies that a decision maker's ordinal preference function can be written in an additive form. When there are three or more attributes, deterministic additive independence is implied when the indifference curves between each pair of attributes do not change with the levels of the remaining attributes.

We provide a new interpretation of deterministic additive independence in terms of the decision maker's marginal value reduction coefficient. Not only is this interpretation necessary and sufficient for an additive ordinal preference attitude, but also it leads to an important theorem for assessment in Section 4.1: we show that any additive value function can be expressed entirely in terms of the decision maker's marginal value reduction coefficients and some scaling constants that are very easy to assess.

The three independence conditions in Chapter 3 form a simple hierarchy: risk additive independence implies mutual utility independence; mutual utility independence implies deterministic additive independence.

Chapters 2 and 3 stand together as a powerful technology for interpreting any twice differentiable multiattribute preference function. We assemble the components of this technology for the purpose of assessment in Chapter 4: the assessment protocol is summarized in a flow chart, and the practical procedure is illustrated by example. These methods are limited to preference attitudes for which the value function can be written in an additive form.

Existing procedures for modeling multiattribute outcomes and assessing preferences over them are far from complete: we suggest two areas for important research contributions in Chapter 5.

1.3 Related Work

The literature on multiattribute preferences is voluminous. Boyd (1970) presents a solid foundation from which to view multiattribute preferences without restrictive assumptions. Pollard (1969) suggests using periodic consumption decisions as a method for gaining insight into joint time/risk decision-making. Barrager (1975), extending the ideas presented by Boyd and Pollard, defines a "coefficient of variation aversion" as a tool for structuring and analyzing preferences over consumption vectors. Barrager's coefficient and the risk aversion coefficient defined by Pratt (1964) inspired the development of preference measures in this dissertation.

Richard (1975) identifies a cross-dimensional aspect of risk aversion and appropriately defines it as "multiattribute risk aversion." His definition led us to derive a multiattribute risk aversion coefficient as a measure of this phenomenon.

Fishburn and Keeney (1974) identify some independence conditions that are important for limiting the functional expression of a multiattribute preference attitude. We use three of these independence conditions in Chapter 3. Howard (1974) discusses a delta property for risk preference in one dimension, which inspired our own development of delta properties for multiattribute risk preference.

Keeney (1969, 1971, 1972, 1973, 1974, and 1975) and Raiffa (1969) provide detailed methods for interpreting and assessing multiattribute

utility functions using utility independence. Keeney and Raiffa (1975) summarizes the bulk of this literature, contains a good introduction to deterministic preferences, and includes an extensive bibliography. The relationships between deterministic and risk preferences, however, are mostly undeveloped in this book. As a result, their multiattribute risk preference assessment procedures require the inference of deterministic preferences from probabilistic questions.

In contrast, this dissertation follows the traditional approach of Stanford decision analysts: generally, deterministic preferences are encoded first, and then risk preference is imposed over the more fundamental deterministic preference structure.

Chapter 2

SIX LOCAL MEASURES OF MULTIATTRIBUTE PREFERENCES

2.1 Introduction to Preference Functions

In this dissertation, a preference function is a mathematical expression of a decision maker's preferences over a set of possible outcomes. Except where explicitly stated otherwise, we assume that each outcome can be described by a vector of real numbers $\underline{x} = (x_1, \ldots, x_n)$ such that more in any dimension is preferred to less. In this section we discuss three types of preference functions (value functions, numeraire functions, and utility functions) and frame them in the skeleton of a procedure for encoding a decision maker's preferences.

A value function $v(\underline{x})$ is a scalar valued function with the properties: if $v(\underline{x}^1) > v(\underline{x}^2)$, then $\underline{x}^1$ is preferred to $\underline{x}^2$; if $v(\underline{x}^1) = v(\underline{x}^2)$, then $\underline{x}^1$ is indifferent to $\underline{x}^2$. Thus, $v(\cdot)$ is an ordinal preference function, which provides a simple ranking of outcome vectors $\underline{x}$.

By setting the value function equal to a constant

$$v(\underline{x}) = c$$

we define an indifference surface, a set of points among which the decision maker is indifferent. Since $v(\underline{x})$ is a simple ordinal ranking, the indifference surfaces and their ordering are all the information that it contains. In two dimensions, indifference surfaces are usually curves: a family of indifference curves is shown in Figure 2.1.

We shall say that a function $f(\cdot)$ is increasing if $f(\bar{x}) > f(\hat{x})$ whenever $\bar{x} > \hat{x}$. Let $f(\cdot)$ be an increasing function. Then, since

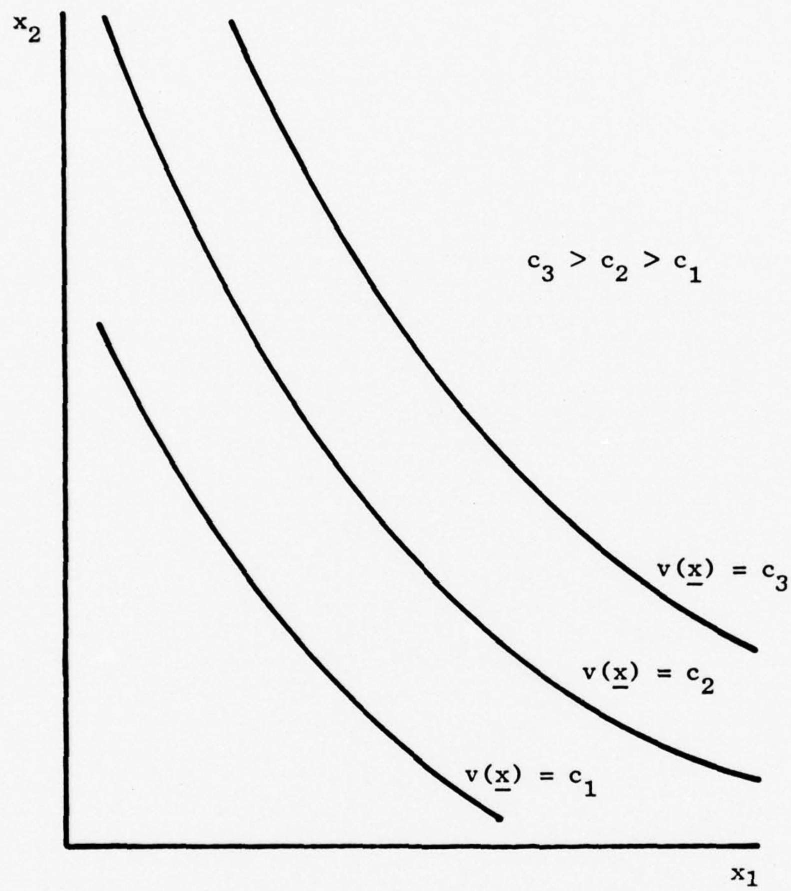


Figure 2.1. INDIFFERENCE CURVES.

the value function $f(v(\underline{x}))$ can be seen to have the same indifference surfaces and ordering as $v(\underline{x})$, a value function is unique up to an increasing transformation.

Also, we shall say that a value function is <u>additive</u> if some increasing transformation of it can be written in this additive form

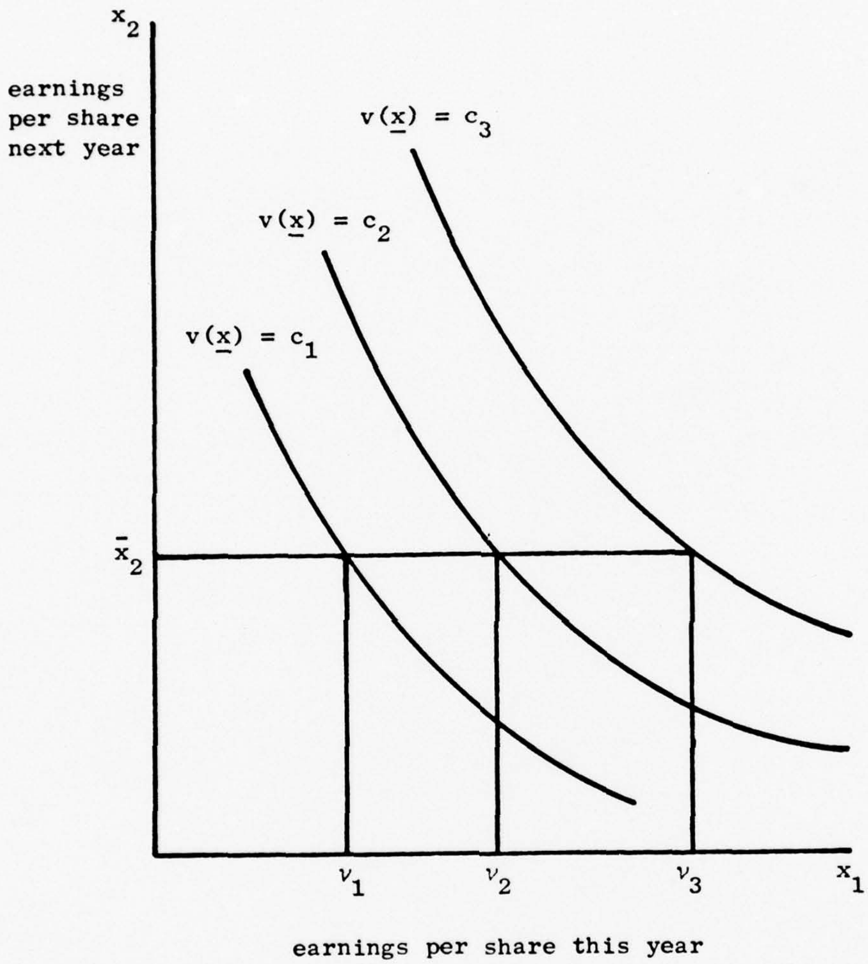
$$v(\underline{x}) = v_1(x_1) + \ldots + v_n(x_n)$$

A <u>numeraire function</u> $\nu(\underline{x})$ is a value function that is defined so the quantity ν is easy for the decision maker to interpret. In particular, a <u>single-attribute numeraire</u> ν is defined implicitly as a function of $\underline{x}$ by

$$v\left(\nu(\underline{x}),\bar{x}_2, \ldots, \bar{x}_n\right) = v(\underline{x})$$

where $\bar{x}_2,\ldots,\bar{x}_n$ are fixed nominal levels. Note that $\nu(\underline{x})$ expresses exactly the same preference ordering as $v(\underline{x})$; the only difference is that v is arbitrary (up to an increasing transformation), whereas ν is easily interpreted as an amount of the first attribute with the other attributes fixed at nominal levels.

For example, indifference curves are plotted in Figure 2.2 that could represent a firm's tradeoffs between earnings per share this year and earnings per share next year. Defining the single-attribute numeraire as above, we can replace the arbitrary indifference curve indexing v with an intuitive indexing ν, which is easily interpreted as an amount of earnings per share this year with earnings per share next year held fixed at the nominal level $\bar{x}_2$.



$$v(\nu_1, \bar{x}_2) = c_1$$
$$v(\nu_2, \bar{x}_2) = c_2$$
$$v(\nu_3, \bar{x}_2) = c_3$$

Figure 2.2. SINGLE-ATTRIBUTE NUMERAIRE.

Similarly, it is sometimes convenient to define a diagonal numeraire by

$$v\left(\nu(\underline{x}),\ \ldots,\ \nu(\underline{x})\right) = v(\underline{x})$$

In Figure 2.3 the diagonal numeraire indexes each indifference curve by its constant annual earnings per share.

We shall use $\{\underline{x}|\mathcal{S}\}$ to denote either a discrete or continuous lottery over $\underline{x}$ given the state of information $\mathcal{S}$. In this notation then, $\{\underline{x}|\mathcal{S}_1\}$ and $\{\underline{x}|\mathcal{S}_2\}$ are two different lotteries over $\underline{x}$. The symbol $\int$ denotes a summation or integration operator as appropriate.

A utility function $u(\underline{x})$ is a scalar valued function with the properties: if

$$\int_{\underline{x}} \{\underline{x}|\mathcal{S}_1\}\ u(\underline{x}) > \int_{\underline{x}} \{\underline{x}|\mathcal{S}_2\}\ u(\underline{x})$$

then lottery $\{\underline{x}|\mathcal{S}_1\}$ is preferred to lottery $\{\underline{x}|\mathcal{S}_2\}$; if

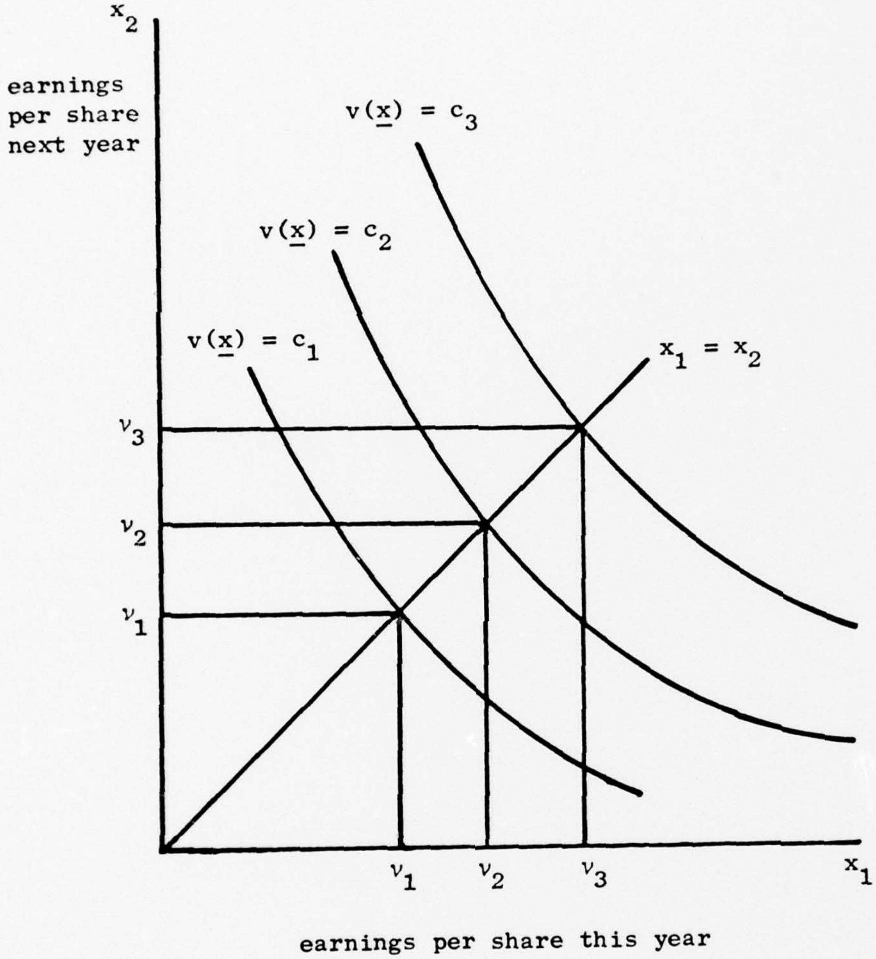
$$\int_{\underline{x}} \{\underline{x}|\mathcal{S}_1\}\ u(\underline{x}) = \int_{\underline{x}} \{\underline{x}|\mathcal{S}_2\}\ u(\underline{x})$$

then lottery $\{\underline{x}|\mathcal{S}_1\}$ is indifferent to lottery $\{\underline{x}|\mathcal{S}_2\}$. In words, a utility function is used to rank lotteries, and the utility of a lottery is the expected utility of its prizes.

If a and b are constants with $a > 0$, it can be seen from the above definition that $\hat{u}(\underline{x})$ where

$$\hat{u}(\underline{x}) = au(\underline{x}) + b$$

implies the same ranking of lotteries as $u(\underline{x})$. Thus, a utility function is unique up to a positive linear transformation.



$$v(\nu_1, \nu_1) = c_1$$

$$v(\nu_2, \nu_2) = c_2$$

$$v(\nu_3, \nu_3) = c_3$$

Figure 2.3. DIAGONAL NUMERAIRE.

In the special case of no uncertainty, lotteries $\{\underline{x}|\delta_1\}$ and $\{\underline{x}|\delta_2\}$ have deterministic outcomes. Defining these outcomes as $\underline{x}^1$ and $\underline{x}^2$, respectively, the definitions for utility function and value function coincide. Therefore, <u>any utility function is a value function</u>, but not vice versa.

A useful approach for constructing a utility function over multiattribute outcomes $\underline{x}$ is called the <u>two-step procedure</u>: first, we assess the decision maker's ordinal preference function $v(\underline{x})$ and from it derive an appropriate numeraire $\nu(\underline{x})$; second, we assess risk preference over the one-dimensional numeraire ν. This yields a multiattribute utility function $u(\nu(\underline{x}))$.

The two-step procedure is illustrated in Figure 2.4 and is used in the example of Section 2.6. Some specific assessment techniques for implementing it under more general conditions are introduced in Chapter 4.

If a multiattribute utility function is assessed without using the two-step procedure, it is usually written without any explicit dependence on a numeraire as $\hat{u}(\underline{x})$. It is worth noting that from this form we can still construct an equivalent multiattribute utility function of the form $u(\nu(\underline{x}))$. To do this we observe that since $\hat{u}(\underline{x})$ is a utility function it is also a value function. So, letting

$$v(\underline{x}) = \hat{u}(\underline{x})$$

we proceed to define a numeraire function $\nu(\underline{x})$ in the manner described above. Then, we look for a function $u(\cdot)$ such that

$$u\left(\nu(\underline{x})\right) = a\hat{u}(\underline{x}) + b \qquad \text{where} \quad a > 0$$

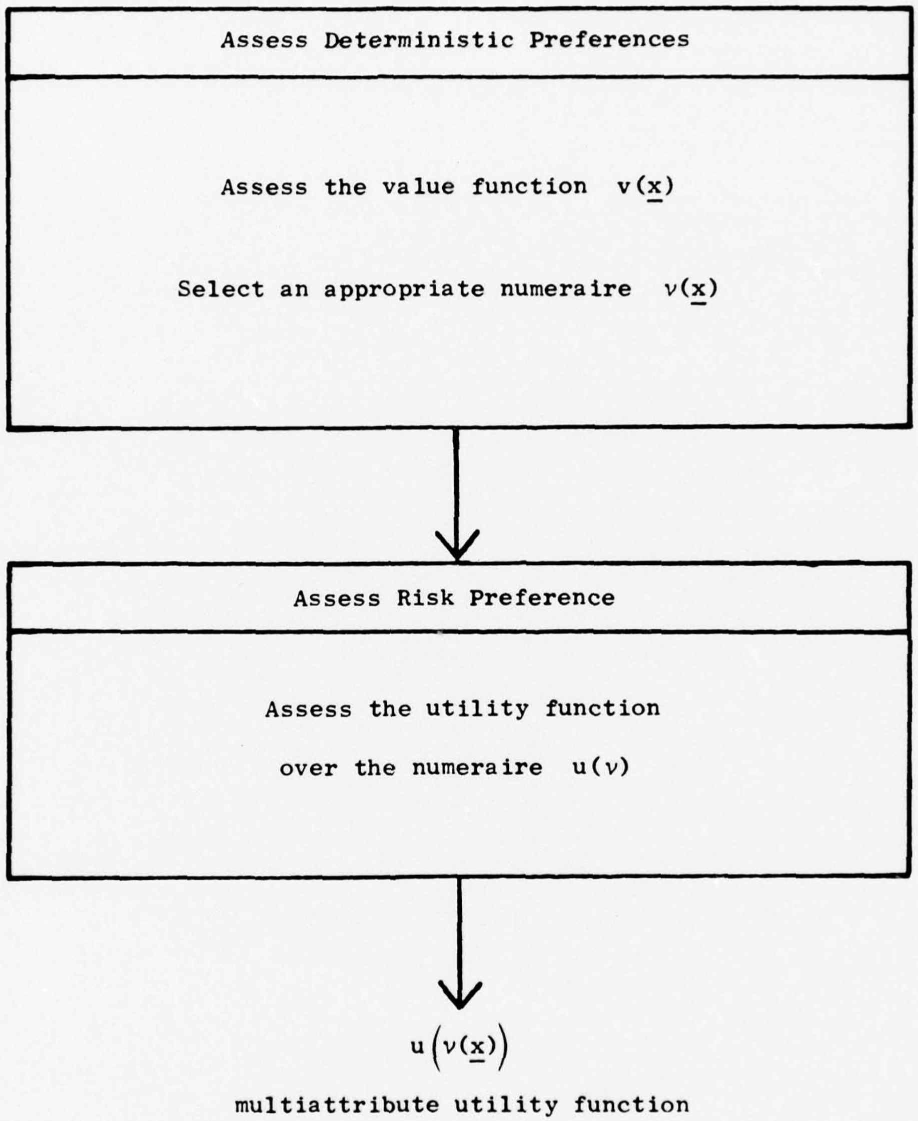


Figure 2.4. A TWO-STEP PROCEDURE FOR ASSESSING MULTIATTRIBUTE RISK PREFERENCE.

For example, in Section 3.4 we present some behavioral conditions for inferring directly that a decision maker's utility function must be a product-of-exponentials

$$\hat{u}(\underline{x}) = -e^{-\gamma_1 x_1 - \cdots - \gamma_n x_n} \qquad \text{where} \quad \gamma_i > 0$$

To construct a single-attribute numeraire, initially treat $\hat{u}(\underline{x})$ as a value function. Realizing that ordinal preferences are preserved under the increasing transformation

$$f(y) = -\ln(-y) \qquad \text{where} \quad y < 0$$

we can write

$$v(\underline{x}) = f\left(\hat{u}(\underline{x})\right)$$

$$= \gamma_1 x_1 + \cdots + \gamma_n x_n$$

Defining the single-attribute numeraire

$$\gamma_1 \nu(\underline{x}) + \gamma_2 \bar{x}_2 + \cdots + \gamma_n \bar{x}_n = \gamma_1 x_1 + \cdots + \gamma_n x_n$$

$$\nu(\underline{x}) = \frac{1}{\gamma_1} (\gamma_1 x_1 + \cdots + \gamma_n x_n - \gamma_2 \bar{x}_2 - \cdots - \gamma_n \bar{x}_n)$$

Now, if we let

$$u\left(\nu(\underline{x})\right) = -e^{-\gamma_1 \nu(\underline{x})}$$

$$u\left(\nu(\underline{x})\right) = -e^{-\gamma_1 x_1 - \cdots - \gamma_n x_n + \gamma_2 \bar{x}_2 + \cdots + \gamma_n \bar{x}_n}$$

$$= a\hat{u}(\underline{x}) \qquad \text{where} \quad a > 0$$

This shows that even when the utility function is initially expressed in the form $\hat{u}(\underline{x})$, we can derive an equivalent utility function of the form $u(\nu(\underline{x}))$.

The form $u(\nu(\underline{x}))$ allows us to define a natural one-dimensional certain equivalent for multiattribute lotteries. For any lottery $\{\underline{x}|\mathcal{S}\}$ and multiattribute utility function $u(\nu(\underline{x}))$, we define the <u>numeraire certain equivalent</u> $\tilde{\nu}$ by

$$u(\tilde{\nu}) = \int_{\underline{x}} \{\underline{x}|\mathcal{S}\}\, u\left(\nu(\underline{x})\right)$$

In words, the utility of the numeraire certain equivalent is equal to the expected utility of the lottery $\{\underline{x}|\mathcal{S}\}$. The numeraire certain equivalent $\tilde{\nu}$ represents the entire indifference surface of outcomes $\underline{x}$ such that

$$\tilde{\nu} = \nu(\underline{x})$$

If $\underline{x}$ is a cash flow, for example, then $\tilde{\nu}$ could be the net present value of all certain cash flows that are indifferent to $\{\underline{x}|\mathcal{S}\}$.

Boyd (1970) discusses the two-step procedure in greater detail and presents axioms guaranteeing the existence of a value function. The normative axioms justifying utility functions are introduced by VonNeumann and Morgenstern (1944), refined for decision-making by Savage (1954), and presented in a particularly clear form by Howard (1974).

Beyond the axioms underlying preference functions, we assume throughout that each preference function is continuous and twice differentiable. Also, we assume that each attribute and numeraire is selected in a way that more is preferred to less.

2.2 Ordinal Preference Measures

Defining local measures of preference attitude is important for two reasons: first, these measures provide simple preference interpretations near any point; second, they can help generate specific functional forms and provide natural questions for assessing parameters.

In this section we define three local measures of ordinal preference attitude: the marginal rate of substitution; the marginal value reduction coefficient; and the substitution aversion coefficient.

Marginal Rate of Substitution

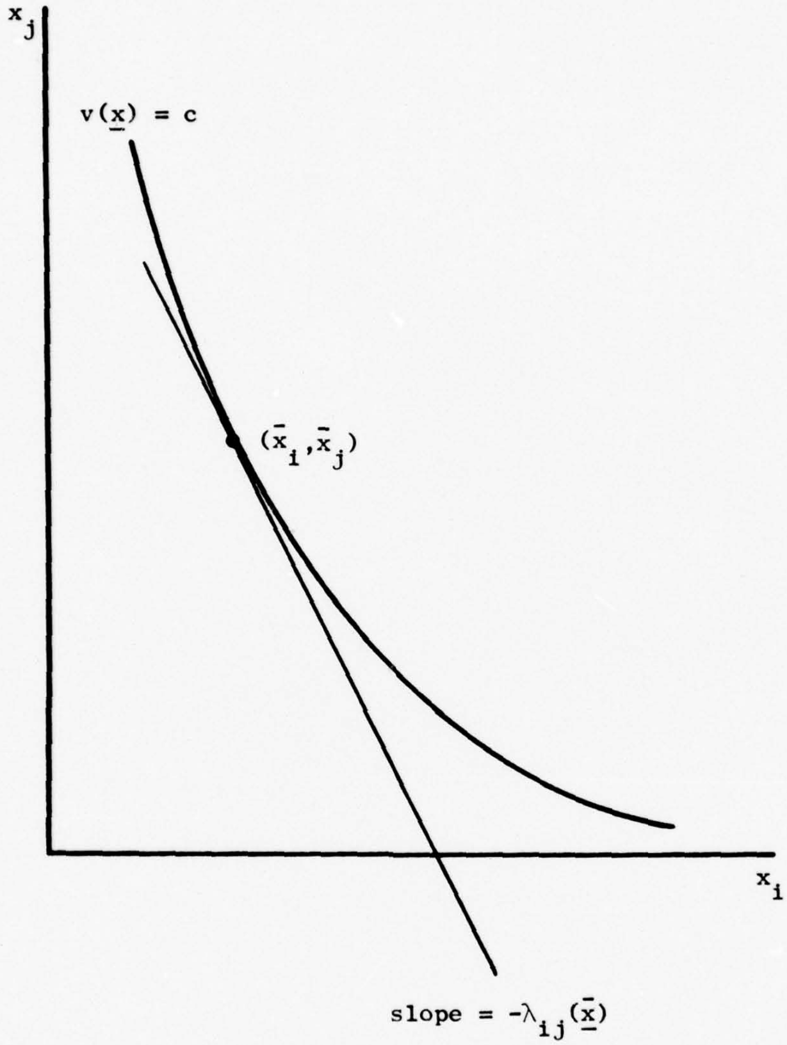
The marginal rate of substitution is a well-known first order approximation of an indifference curve. For example, if a peach-lover is viewing his tradeoffs between dollars and peaches, then, at each level of wealth in dollars and peaches, the marginal rate of substitution is the dollar price that he is just willing to pay for an additional peach.

Mathematically, the marginal rate of substitution is the negative of the slope of an indifference curve at a point and is interpreted in Figure 2.5. Let $\underline{x}_{ij} = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1},x_{j+1},\ldots,x_n)$. Then an indifference curve between x_i and x_j is implied by setting the value function equal to a constant and fixing $\underline{x}_{ij}$ at some nominal level $\bar{\underline{x}}_{ij}$:

$$v(\underline{x}) = c \qquad \text{where} \quad \underline{x}_{ij} = \bar{\underline{x}}_{ij}$$

Differentiating with respect to x_i

$$\frac{\partial v}{\partial x_i} + \frac{\partial v}{\partial x_j}\frac{dx_j}{dx_i} = 0$$



$\lambda_{ij}(\bar{\underline{x}})$ is the answer to, "Starting at $\bar{\underline{x}}$, how much x_j are you just willing to give up in order to get an additional unit of x_i?"

Figure 2.5. MARGINAL RATE OF SUBSTITUTION ($\underline{x}_{ij} = \bar{\underline{x}}_{ij}$).

$$\frac{dx_j}{dx_i} = - \frac{\frac{\partial v}{\partial x_i}}{\frac{\partial v}{\partial x_j}}$$

<u>Definition 2.1</u>. Marginal rate of substitution $\lambda_{ij}(\underline{x})$. Let $v(\underline{x})$ be a value function. Then

$$\lambda_{ij}(\underline{x}) = \frac{\frac{\partial v}{\partial x_i}}{\frac{\partial v}{\partial x_j}}$$

To simplify notation we shall use $\lambda = \lambda_{ij}(\underline{x})$.

As a fundamental property of a decision maker's preference attitude, λ is unique under any increasing *transformation* of the value function. To see this, let

$$\hat{v}(\underline{x}) = f\Big(v(\underline{x})\Big)$$

where $f(\cdot)$ is differentiable and increasing. Then

$$\hat{\lambda} = \frac{\frac{\partial f}{\partial v}\,\frac{\partial v}{\partial x_i}}{\frac{\partial f}{\partial v}\,\frac{\partial v}{\partial x_j}} = \frac{\frac{\partial v}{\partial x_i}}{\frac{\partial v}{\partial x_j}} = \lambda$$

Also note that $\lambda_{ij}(\underline{x})$, which has units x_j/x_i, can be interpreted as the x_j price that the decision maker is just willing to pay for an additional unit of x_i.

Marginal Value Reduction Coefficient

Suppose that our peach-lover receives a crate of peaches as an anonymous gift. Then, even with his affinity for peaches, we expect that the dollar price that he is willing to pay for an additional peach is less than it was before.

The marginal value reduction coefficient captures this notion that as a decision maker becomes wealthier in one attribute the marginal price that he is willing to pay for an additional unit of it usually declines. As shown in Figure 2.6, this coefficient measures the reduction in marginal value that follows from increasing wealth in an attribute.

Definition 2.2. Marginal value reduction coefficient $z_{ij}(\underline{x})$.

$$z_{ij}(\underline{x}) = -\frac{\frac{\partial\lambda}{\partial x_i}}{\lambda}$$

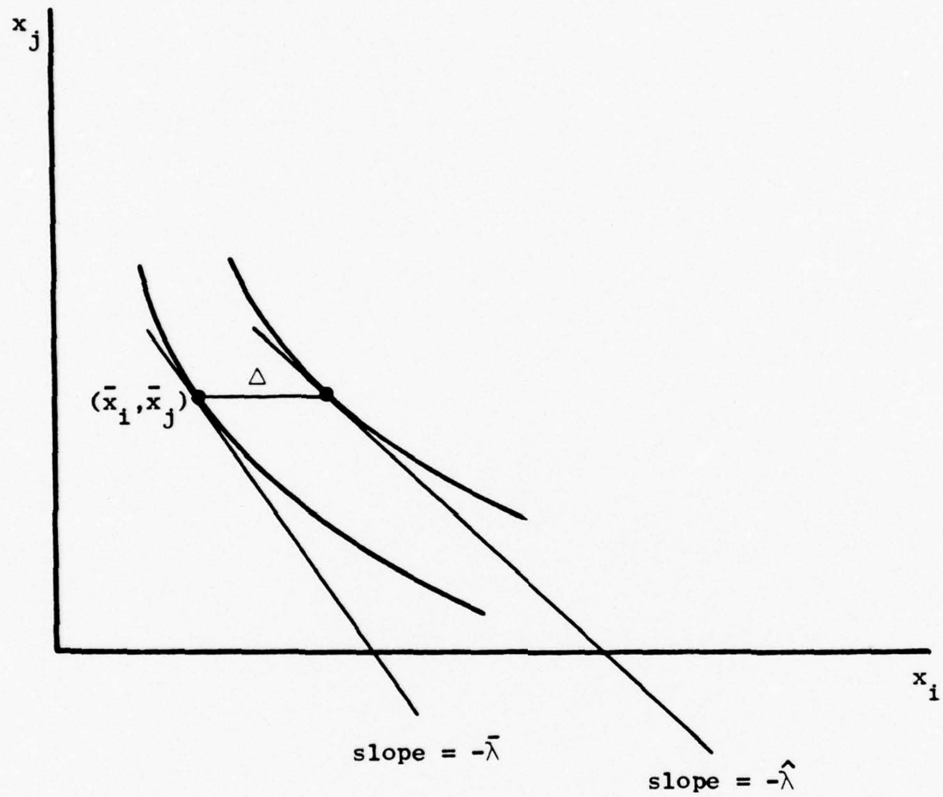
In words, $z_{ij}(\underline{x})$ is the percent decrease in the marginal price λ per unit increase of wealth in x_i.

Note that $z_{ij}(\underline{x})$ has units $1/x_i$ and measures an aspect of decision maker preferences that depends only on the marginal rate of substitution λ. Therefore, like λ, $z_{ij}(\underline{x})$ is unique under any increasing transformation of the value function.

In terms of the value function,

$$\lambda = \frac{\frac{\partial v}{\partial x_i}}{\frac{\partial v}{\partial x_j}}$$

$$z_{ij}(x) = - \frac{\frac{\partial\lambda}{\partial x_i}}{\lambda}$$



If $z_{ij}(\underline{x}) > 0$, then $\bar{\lambda} > \hat{\lambda}$

Figure 2.6. MARGINAL VALUE REDUCTION COEFFICIENT $(\underline{x}_{ij} = \underline{\bar{x}}_{ij})$.

$$\frac{\partial\lambda}{\partial x_i} = \frac{\dfrac{\partial v}{\partial x_j}\dfrac{\partial^2 v}{\partial x_i^2} - \dfrac{\partial v}{\partial x_i}\dfrac{\partial^2 v}{\partial x_i \partial x_j}}{\left(\dfrac{\partial v}{\partial x_j}\right)^2}$$

$$= \lambda \frac{\dfrac{\partial^2 v}{\partial x_i^2}}{\dfrac{\partial v}{\partial x_i}} - \lambda \frac{\dfrac{\partial^2 v}{\partial x_i \partial x_j}}{\dfrac{\partial v}{\partial x_j}}$$

$$z_{ij}(\underline{x}) = - \frac{\dfrac{\partial^2 v}{\partial x_i^2}}{\dfrac{\partial v}{\partial x_i}} + \frac{\dfrac{\partial^2 v}{\partial x_i \partial x_j}}{\dfrac{\partial v}{\partial x_j}} \qquad (2.1)$$

Deriving the marginal value reduction coefficient for a particular preference function is usually easiest using this equation. Note that if $v(\underline{x})$ is additive, $z_{ij}(\underline{x})$ depends only on x_i; the converse is also true and is proven in Chapter 3.

For assessment, it will be useful to estimate these coefficients from readily available data, such as $\bar{\underline{x}}$, $\bar{\lambda}$, $\hat{\lambda}$, and Δ in Figure 2.6. Let $\underline{x}_{\bar{i}} = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$. To estimate $z_{ij}(\bar{\underline{x}})$ we assume that

$$v(\underline{x}) = v_i(x_i) + w_i(\underline{x}_{\bar{i}})$$

Then

$$\lambda_{ij}(\underline{x}) = \frac{v_i'(x_i)}{w_i'(\underline{x}_{\bar{i}})} \qquad \text{where} \quad v_i'(x_i) = \frac{dv_i}{dx_i}$$

$$w_i'(\underline{x}_{\bar{i}}) = \frac{\partial w_i}{\partial x_j}, \quad j \neq i$$

$$\bar{\lambda} = \frac{v_i'(\bar{x}_i)}{w_i'(\bar{\underline{x}}_i)}$$

$$\hat{\lambda} = \frac{v_i'(\bar{x}_i + \Delta)}{w_i'(\bar{\underline{x}}_i)}$$

$$\frac{\bar{\lambda}}{\hat{\lambda}} = \frac{v_i'(\bar{x}_i)}{v_i'(\bar{x}_i + \Delta)} \tag{2.2}$$

Now, if

$$v_i(x_i) = \begin{cases} -a_i\, e^{-\gamma_i x_i} & \text{for } \gamma_i \neq 0 \\ a_i x_i & \text{for } \gamma_i = 0 \end{cases}$$

then it is easy to show using (2.1) that

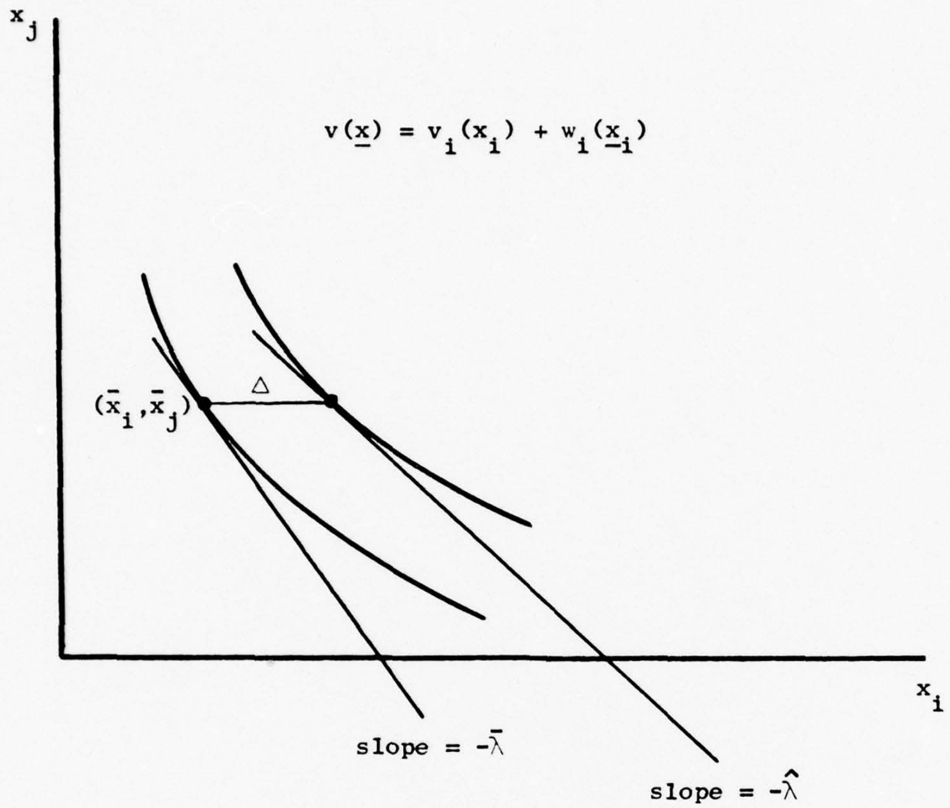
$$z_{ij}(\underline{x}) = \gamma_i \qquad \text{for } j \neq i$$

For $\gamma_i \neq 0$

$$\frac{\bar{\lambda}}{\hat{\lambda}} = \frac{a_i\gamma_i\, e^{-\gamma_i x_i}}{a_i\gamma_i\, e^{-\gamma_i (x_i + \Delta)}} = e^{\gamma_i \Delta}$$

$$z_{ij}(\bar{\underline{x}}) = \frac{1}{\Delta} \ln \frac{\bar{\lambda}}{\hat{\lambda}}$$

If $\gamma_i = 0$, then $\bar{\lambda} = \hat{\lambda}$, and this equation still holds. Thus, given $\bar{\underline{x}}$, $\bar{\lambda}$, $\hat{\lambda}$, and Δ, this formula provides an exponential estimate for $z_{ij}(\bar{\underline{x}})$. Similarly, a power form estimate for $z_{ij}(\bar{\underline{x}})$ is derived in Appendix H. Both estimates are illustrated in Figure 2.7.



<u>exponential estimate</u>

$$v_i(x_i) = \begin{cases} -a_i\, e^{-\gamma_i x_i} & \gamma_i \neq 0 \\ a_i x_i & \gamma_i = 0 \end{cases}$$

$$z_{ij}(\underline{\bar{x}}) = \gamma_i$$

$$= \frac{1}{\Delta} \ln \frac{\bar{\lambda}}{\hat{\lambda}}$$

<u>power form estimate ($x_i > 0$)</u>

$$v_i(x_i) = \begin{cases} -a_i x_i^{-\alpha_i} & \alpha_i \neq 0 \\ a_i \ln x_i & \alpha_i = 0 \end{cases}$$

$$z_{ij}(\underline{\bar{x}}) = \frac{1 + \alpha_i}{\bar{x}_i}$$

$$= \frac{\ln \frac{\bar{\lambda}}{\hat{\lambda}}}{\bar{x}_i \ln \left(1 + \frac{\Delta}{\bar{x}_i}\right)}$$

Figure 2.7. TWO METHODS FOR ESTIMATING THE MARGINAL VALUE REDUCTION COEFFICIENT ($\underline{x}_{-ij} = \underline{\bar{x}}_{-ij}$).

The exponential estimate is also useful for interpreting numerical values of the marginal value reduction coefficient. This is easiest in terms of its reciprocal.

<u>Definition 2.3</u>. Marginal value preservation tolerance $\zeta_{ij}(\underline{x})$.

$$\zeta_{ij}(\underline{x}) = \frac{1}{z_{ij}(\underline{x})}$$

If a decision maker has a large tolerance for the preservation of marginal value, then the marginal value λ does not decline much even for a large increase in wealth Δ. On the other hand, if the decision maker has a small tolerance for the preservation of marginal value, then the marginal value λ declines rapidly as Δ increases.

If $v_i(x_i)$ is exponential in Figure 2.7, then

$$\zeta_{ij}(\underline{\bar{x}}) = \frac{\Delta}{\ln \frac{\bar{\lambda}}{\hat{\lambda}}}$$

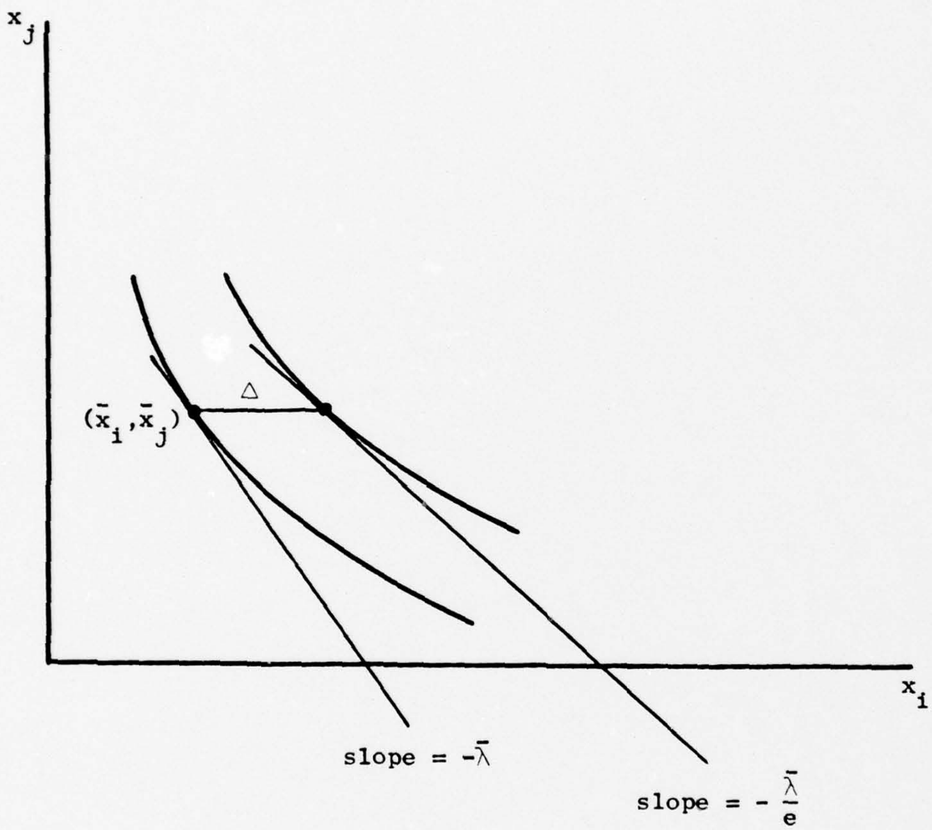
Further, if $\hat{\lambda} = \bar{\lambda}/e$, then

$$\zeta_{ij}(\underline{\bar{x}}) = \Delta$$

Therefore, for an exponential $v_i(x_i)$, the marginal value preservation tolerance is the increase in x_i that decreases the marginal value from λ to λ/e. This is illustrated in Figure 2.8. It is often convenient to think of numerical values of the marginal value preservation tolerance in terms of this figure.

$$v(\underline{x}) = v_i(x_i) + w_i(\underline{x}_{-i})$$

$$v_i(x_i) = -a_i\, e^{-\gamma_i x_i}$$



marginal value preservation tolerance

$$\zeta_{ij}(\bar{\underline{x}}) = \Delta$$

Figure 2.8. EXPONENTIAL INTERPRETATION OF THE MARGINAL VALUE PRESERVATION TOLERANCE ($\underline{x}_{-ij} = \bar{\underline{x}}_{-ij}$).

If $v(\underline{x})$ is an additive value function over a consumption vector $\underline{x}$, then $z_{ij}(\underline{x})$ reduces to the "coefficient of variation aversion" defined by Barrager (1975). Barrager's coefficient is useful for interpreting consumption preferences, but the marginal value reduction coefficient is more general: it is defined for any value function, additive or nonadditive; it is a fundamental property of ordinal preferences that is unique under any increasing transformation of the value function; it is easy to interpret for any multiattribute outcome vector $\underline{x}$, including, of course, a consumption vector as a special case.

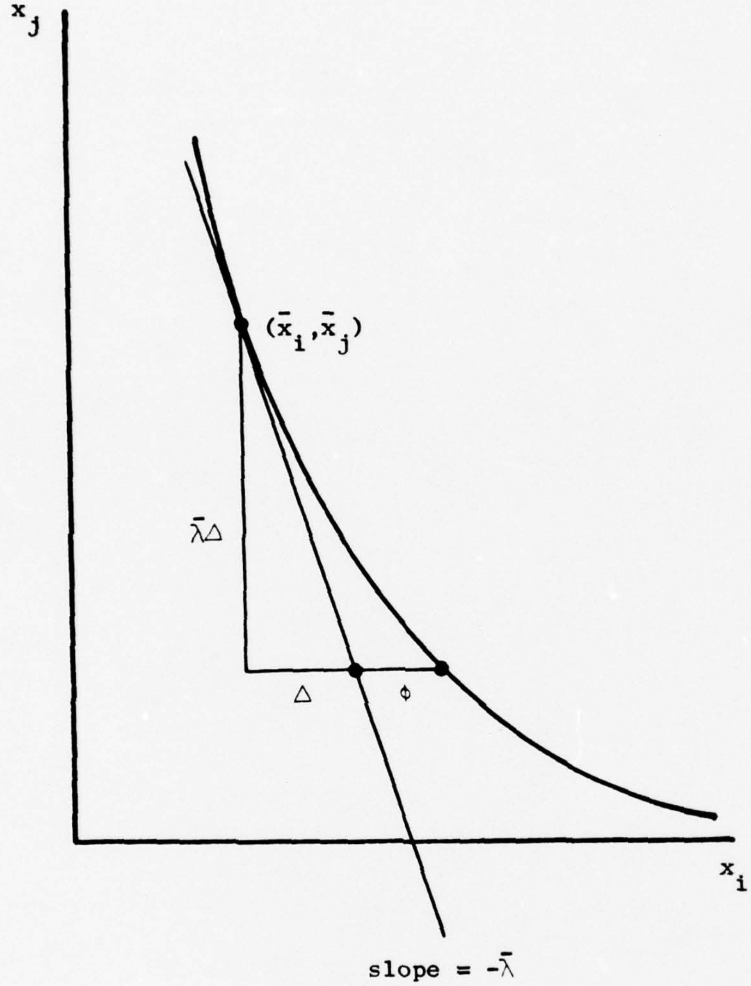
Substitution Aversion Coefficient

The substitution aversion coefficient is a second-order measure of the convexity of an indifference curve and has a natural graphical interpretation. See Figure 2.9.

Starting at $(\bar{x}_i, \bar{x}_j)$, we ask the decision maker whether he would like to move to the new point $(\bar{x}_i + \Delta, \bar{x}_j - \bar{\lambda}\Delta)$. If he refuses our offer, as he will if his indifference curve is convex, we say that he is averse to the substitution of x_i for x_j at the marginal rate. We can then begin to measure his aversion to substitution by asking, "How many additional units of x_i, ϕ, must we give you so that you are indifferent between $(\bar{x}_i, \bar{x}_j)$ and $(\bar{x}_i + \Delta + \phi, \bar{x}_j - \bar{\lambda}\Delta)$?" Then ϕ, which we call the substitution premium, is a measure of the decision maker's aversion to substitution corresponding to the increment Δ. Approximating ϕ with a Taylor expansion of the indifference curve motivates our definition of the substitution aversion coefficient.

As before, we define an indifference curve for x_i and x_j by

$$v(x) = c \qquad \text{where } \underline{x}_{\overline{ij}} = \bar{\underline{x}}_{\overline{ij}}$$



$$s_{ij}(x) = -\frac{\frac{d\lambda}{dx_i}}{\lambda}$$

$$\phi \approx \frac{1}{2}\, s_{ij}(\bar{\underline{x}})\, \Delta^2$$

Figure 2.9. SUBSTITUTION AVERSION COEFFICIENT ($\underline{x}_{-ij} = \bar{\underline{x}}_{-ij}$).

Consider that this equation implicitly defines x_i as a function of x_j. Using $x_i = g(x_j)$ and differentiating,

$$\frac{\partial v}{\partial x_j} + \frac{\partial v}{\partial x_i} g'(x_j) = 0$$

$$g'(x_j) = -\frac{\dfrac{\partial v}{\partial x_j}}{\dfrac{\partial v}{\partial x_i}} = -\frac{1}{\lambda}$$

$$g''(x_j) = \frac{1}{\lambda^2} \frac{d\lambda}{dx_j}$$

We now use Taylor's theorem to approximate ϕ by expanding about $\underline{\bar{x}}$ in Figure 2.9.

$$g(\bar{x}_j - \bar{\lambda}\Delta) \approx g(\bar{x}_j) - \bar{\lambda}\Delta g'(\bar{x}_j) + \frac{\bar{\lambda}^2\Delta^2}{2} g''(\bar{x}_j)$$

$$g(\bar{x}_j - \bar{\lambda}\Delta) - g(\bar{x}_j) \approx -\bar{\lambda}\Delta\left(-\frac{1}{\lambda}\right) + \frac{\bar{\lambda}^2\Delta^2}{2} \frac{1}{\bar{\lambda}^2} \left.\frac{d\lambda}{dx_j}\right|_{\underline{\bar{x}}}$$

$$\Delta + \phi \approx \Delta + \frac{\Delta^2}{2} \left.\frac{d\lambda}{dx_j}\right|_{\underline{\bar{x}}}$$

$$\phi \approx \frac{1}{2}\left(\left.\frac{d\lambda}{dx_j}\right|_{\underline{\bar{x}}}\right)\Delta^2 \tag{2.3}$$

Now, in general, when moving along an indifference curve

$$\frac{d\lambda}{dx_j} = \frac{\partial\lambda}{\partial x_j} + \frac{\partial\lambda}{\partial x_i}\frac{dx_i}{dx_j}$$

$$= \frac{\partial\lambda}{\partial x_j} - \frac{1}{\lambda}\frac{\partial\lambda}{\partial x_i}$$

$$\frac{d\lambda}{dx_i} = \frac{\partial\lambda}{\partial x_i} + \frac{\partial\lambda}{\partial x_j}\frac{dx_j}{dx_i}$$

$$= \frac{\partial\lambda}{\partial x_i} - \lambda\frac{\partial\lambda}{\partial x_j} \tag{2.4}$$

Thus,

$$\frac{d\lambda}{dx_j} = -\frac{\frac{d\lambda}{dx_i}}{\lambda} \tag{2.5}$$

<u>Definition 2.4</u>. Substitution aversion coefficient $s_{ij}(\underline{x})$.

$$s_{ij}(\underline{x}) = -\frac{\frac{d\lambda}{dx_i}}{\lambda}$$

where $d\lambda/dx_i$ is given by Equation (2.4) and is just the derivative of λ restricted to the indifference curve.

Substituting (2.5) into (2.3) and using the definition above allows us to rewrite (2.3)

$$\phi \approx \frac{1}{2}\, s_{ij}(\bar{\underline{x}})\, \triangle^2 \tag{2.6}$$

This approximation is exact if x_i is a quadratic function of x_j for the indifference curve in Figure 2.9.

In terms of the value function, $s_{ij}(\underline{x})$ can be written

$$s_{ij}(\underline{x}) = -\frac{\frac{\partial^2 v}{\partial x_i^2}}{\frac{\partial v}{\partial x_i}} - \frac{\frac{\partial v}{\partial x_i}\frac{\partial^2 v}{\partial x_j^2}}{\left(\frac{\partial v}{\partial x_j}\right)^2} + 2\,\frac{\frac{\partial^2 v}{\partial x_i \partial x_j}}{\frac{\partial v}{\partial x_j}}$$

This is shown in Appendix C.

Note that $s_{ij}(\underline{x})$ has units $1/x_i$ and measures an aspect of decision maker preferences that depends only on the marginal rate of substitution λ. Therefore, like λ, $s_{ij}(\underline{x})$ is unique under any increasing transformation of the value function.

The substitution aversion coefficient has many useful interpretations. These begin with the following theorem.

<u>Theorem 2.1</u>. Locally, the indifference curves of a value function are

$$\text{strictly convex if and only if } s_{ij}(\underline{x}) > 0;$$
$$\text{linear if and only if } s_{ij}(\underline{x}) = 0;$$
$$\text{strictly concave if and only if } s_{ij}(\underline{x}) < 0.$$

<u>Proof</u>. By assumption $\partial v/\partial x_i$ and $\partial v/\partial x_j$ are both positive. Thus,

$$\lambda = \frac{\frac{\partial v}{\partial x_i}}{\frac{\partial v}{\partial x_j}} > 0$$

If $s_{ij}(\underline{x}) > 0$, then by definition

$$-\frac{\frac{d\lambda}{dx_i}}{\lambda} > 0$$

$$-\frac{d\lambda}{dx_i} > 0$$

Thus, at $\underline{x}$ the slope of the indifference curve is increasing, which implies strict convexity.

Conversely, strict convexity for a twice differentiable function implies an increasing slope. Thus, $-(d\lambda/dx_i) > 0$, and the substitution aversion coefficient is positive.

The conditions for linearity and strict concavity follow in the same way. Q.E.D.

We interpret this theorem in terms of Figure 2.9. If the indifference curve is convex, then $s_{ij}(\bar{\underline{x}}) > 0$, and the decision maker is averse to substitution at the marginal rate. In this case we shall say that he is <u>substitution averse</u>. If the indifference curve is linear, then $s_{ij}(\bar{\underline{x}}) = 0$, the decision maker has no aversion to substitution at the marginal rate, and we shall say that he is <u>substitution indifferent</u>. Similarly, if the indifference curve is concave, we shall say that the decision maker is <u>substitution preferring</u>.

The following simple relationship between the substitution aversion coefficient and the marginal value reduction coefficients is derived in Appendix C.

$$s_{ij}(\underline{x}) = z_{ij}(\underline{x}) + \lambda z_{ji}(\underline{x})$$

At each point the substitution aversion coefficient is just a weighted average of marginal value reduction coefficients. This knowledge provides us with a fundamental interpretation of the shape of indifference curves.

Clearly, for the decision maker to be substitution preferring ($s_{ij}(\underline{x}) < 0$), one or both of the marginal value reduction coefficients must be negative. A negative marginal value reduction coefficient means that as a decision maker becomes wealthier in an attribute he is

willing to pay <u>more</u> for an additional unit of it. This would be characteristic of a commodity that is addictive in increasing amounts. One example is heroin; another might be cigarettes, if the decision maker is just starting to smoke.

If the decision maker is willing to pay more for an additional cigarette after each one he smokes and more for an additional heroin injection after each one he takes, then his marginal value reduction coefficients are both negative and his tradeoffs between these two attributes are strictly concave. See Figure 2.10.

If the decision maker is trading off heroin and dollars, however, we expect only one increasingly addictive attribute: as he consumes more heroin his dollar price for an additional injection might increase; but as he becomes wealthier in dollars we would not expect him to pay a higher price in heroin for an additional dollar. In this case, the first marginal value reduction coefficient is negative and the second is positive. Thus, his indifference curves could be either concave, linear, or convex, depending on the marginal rate of substitution and the strength of his addiction. These three cases are illustrated in Figure 2.11.

Therefore, the signs of the substitution aversion coefficient and the marginal value reduction coefficients give us important information about the shape of a family of indifference curves.

The reciprocal of the substitution aversion coefficient has units x_i and is often easy to interpret.

<u>Definition 2.5</u>. Substitution tolerance $\sigma_{ij}(\underline{x})$.

$$\sigma_{ij}(\underline{x}) = \frac{1}{s_{ij}(\underline{x})}$$

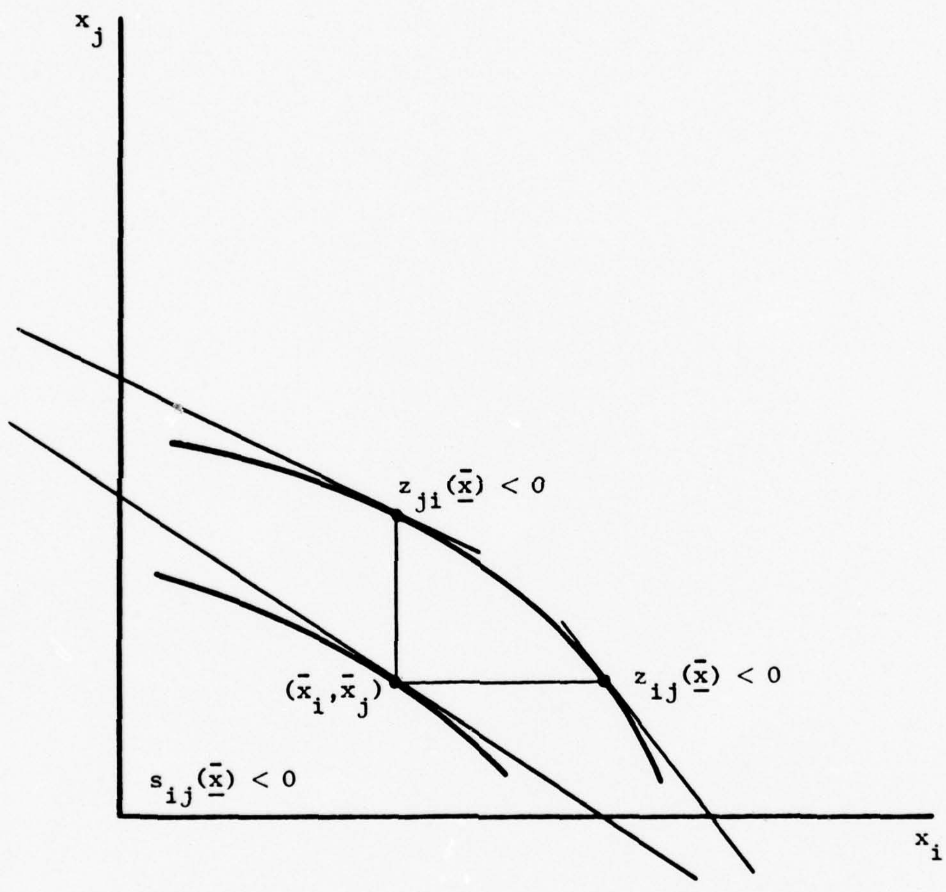


Figure 2.10. TWO ADDICTIVE ATTRIBUTES: CONCAVE INDIFFERENCE CURVES.

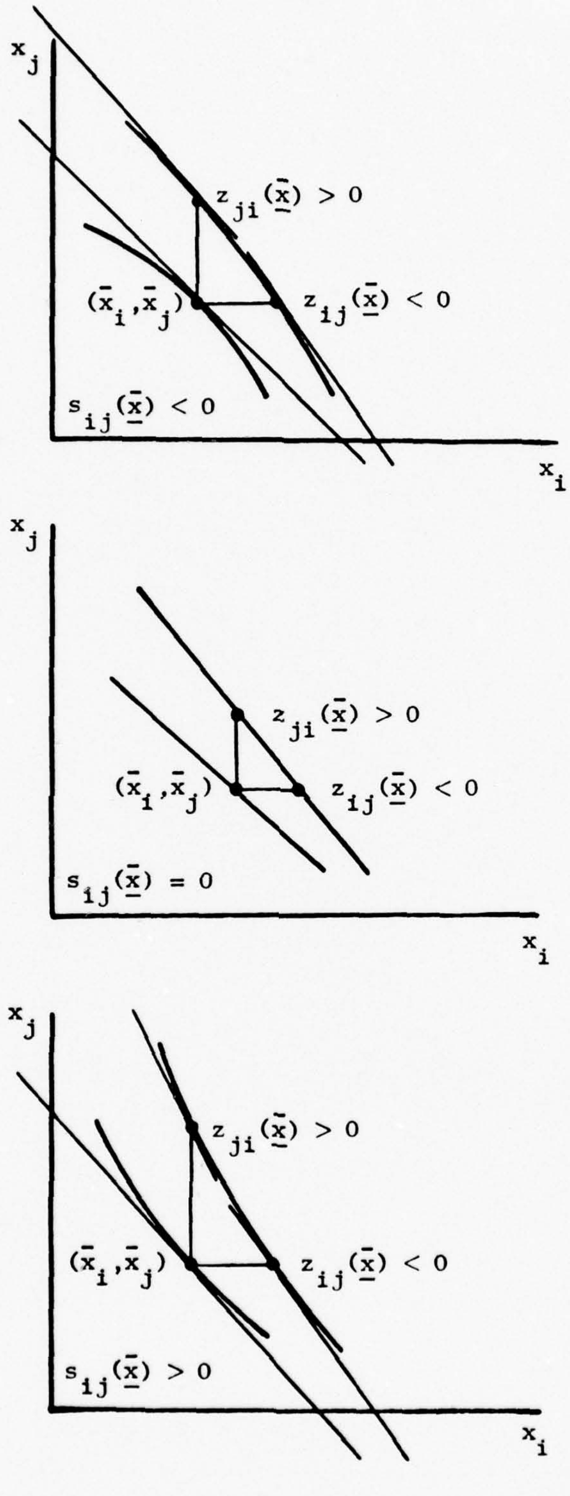


Figure 2.11. INDIFFERENCE CURVES WHEN ONLY ATTRIBUTE x_i IS ADDICTIVE.

If, for example, the decision maker's indifference curves are parallel straight lines, then he has an infinite tolerance for substitution at the marginal rate.

The substitution aversion coefficient is a deterministic analog to Pratt's risk aversion coefficient, which we discuss in the next section. Both coefficients have units $1/x_i$, are equivalent to the negative ratio of the derivative of the slope to the slope, and yield a simple behavioral interpretation through a Taylor expansion (compare Equation (2.6) with Equation (2.9) in the next section).

2.3 Risk Preference Measures

To prepare for discussing risk aversion in a multiattribute context, we first review the well-known measure of risk aversion for one-dimensional outcomes defined by Pratt (1964). Let $u(x)$ be a utility function over a one-dimensional outcome x.

Definition 2.6. Risk aversion coefficient (Pratt) $r(x)$.

$$r(x) = -\frac{\frac{d^2u}{dx^2}}{\frac{du}{dx}}$$

Suppose that the decision maker is facing a lottery $\{x|\mathcal{S}\}$ with mean $\bar{x}$, variance $\overset{\vee}{x}$, and certain equivalent $\tilde{x}$. Then, using a Taylor expansion and the above definition, Pratt derives the following approximation:

$$\tilde{x} \approx \bar{x} - \frac{1}{2}\, r(\bar{x})\, \overset{\vee}{x} \tag{2.9}$$

Thus, the risk aversion coefficient measures the influence of risk aversion on the decision maker's certain equivalent: when $r(x) > 0$, increasing variance $\check{x}$ decreases the certain equivalent $\tilde{x}$. The difference between the mean $\bar{x}$ and certain equivalent $\tilde{x}$ is called the <u>risk premium</u>. It is well-known (see Howard (1971)) that if $u(x)$ is an exponential function and $\{x|\mathcal{S}\}$ is normal, then Equation (2.9) is exact.

Numerical interpretation of the risk aversion coefficient is easier in terms of its reciprocal.

<u>Definition 2.7.</u> Risk tolerance $\rho(x)$.

$$\rho(x) = \frac{1}{r(x)}$$

To interpret the risk tolerance consider the lottery in Figure 2.12. Starting at a nominal wealth level $\bar{x}$, we ask the decision maker to specify an amount Δ such that he is indifferent between staying with certainty at the wealth $\bar{x}$ and participating in the lottery that gives him probability one-half of increasing his wealth by Δ and probability one-half of decreasing his wealth by $\Delta/2$. Then, as indicated in Figure 2.12, Δ is approximately equal to the decision maker's local risk tolerance $\rho(\bar{x})$. Howard (1975) shows that for an exponential utility function this approximation is accurate within four percent:

$$\rho(x) \approx 1.039\,\Delta \approx \Delta$$

Since the exponential utility function is often appropriate within the span of the lottery prizes in Figure 2.12, this lottery interpretation of risk tolerance is quite useful for interpreting and assessing numerical values of the risk aversion coefficient.

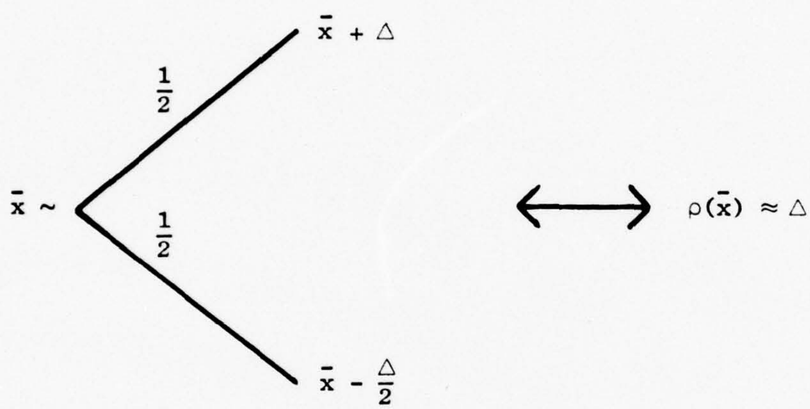


Figure 2.12. INTERPRETATION OF RISK TOLERANCE (EXPONENTIAL UTILITY FUNCTION).

The risk aversion coefficient and its interpretations (Equation (2.9) and Figure 2.12) are also useful in a multiattribute context: the numeraire risk aversion coefficient and single-attribute risk aversion coefficient simply use Pratt's coefficient to measure local risk attitude over important one-dimensional quantities.

Numeraire Risk Aversion Coefficient

Using the form $u(\nu(\underline{x}))$ it is always possible to assess (and view) a multiattribute utility function as a utility function over a one-dimensional numeraire. The following definitions are useful for assessing and analyzing risk preference over the numeraire.

Definition 2.8. Numeraire risk aversion coefficient $r_\nu(\underline{x})$.

$$r_\nu(\underline{x}) = - \left. \frac{\frac{d^2u}{d\nu^2}}{\frac{du}{d\nu}} \right|_{\nu(\underline{x})}$$

To simplify notation, we shall often use $r_\nu = r_\nu(\underline{x})$.

Definition 2.9. Numeraire risk tolerance $\rho_\nu(\underline{x})$.

$$\rho_\nu(\underline{x}) = \frac{1}{r_\nu(\underline{x})}$$

These definitions can be interpreted in Equation (2.9) and Figure 2.12 by simply considering x to be a numeraire outcome.

Single-Attribute Risk Aversion Coefficient

A multiattribute utility function specifies the decision maker's risk preference in any single dimension. The definitions below allow us to look at risk aversion over lotteries in each single attribute while the other attributes are fixed at a known level.

Definition 2.10. Single-attribute risk aversion coefficient $r_i(\underline{x})$.

$$r_i(\underline{x}) = -\frac{\dfrac{\partial^2 u}{\partial x_i^2}}{\dfrac{\partial u}{\partial x_i}}$$

Definition 2.11. Single-attribute risk tolerance $\rho_i(\underline{x})$.

$$\rho_i(\underline{x}) = \frac{1}{r_i(\underline{x})}$$

$r_i(\underline{x})$ and $\rho_i(\underline{x})$ can be interpreted in Equation (2.9) and Figure 2.12 by simply considering x_i outcomes with the other attributes fixed at a known level.

The following derivation was presented by James E. Matheson in an advanced decision analysis lecture. When the multiattribute utility function is in the form $u(v(\underline{x}))$, $r_i(\underline{x})$ can be expressed simply in terms of the numeraire risk aversion coefficient and the derivatives of the numeraire function.

$$\frac{\partial u}{\partial x_i} = \frac{du}{dv}\frac{\partial v}{\partial x_i}$$

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{d^2 u}{d\nu^2}\left(\frac{\partial \nu}{\partial x_i}\right)^2 + \frac{du}{d\nu}\frac{\partial^2 \nu}{\partial x_i^2}$$

$$r_i(\underline{x}) = -\frac{\dfrac{d^2 u}{d\nu^2}\left(\dfrac{\partial \nu}{\partial x_i}\right)^2 + \dfrac{du}{d\nu}\dfrac{\partial^2 \nu}{\partial x_i^2}}{\dfrac{du}{d\nu}\dfrac{\partial \nu}{\partial x_i}}$$

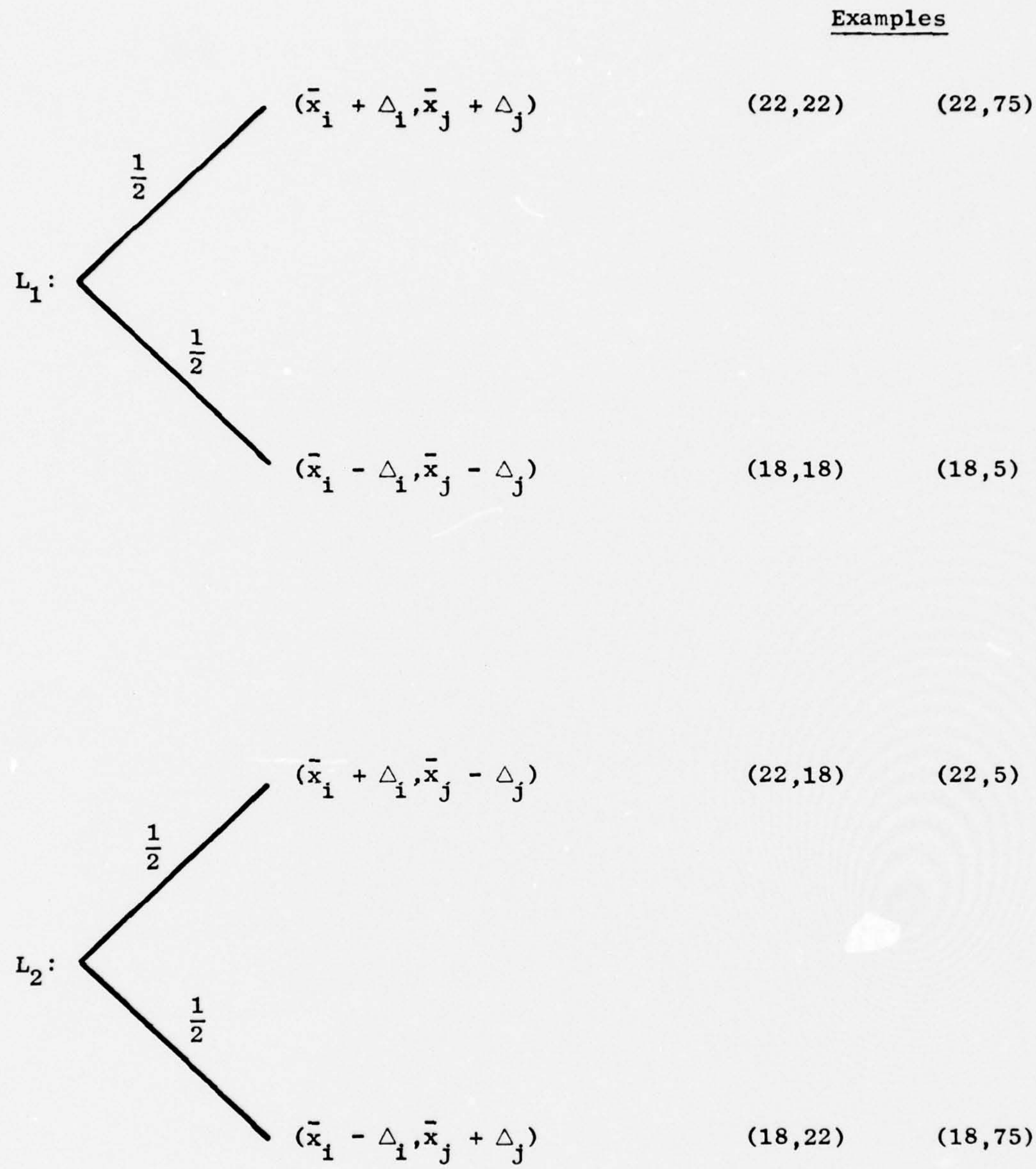
$$= r_\nu \frac{\partial \nu}{\partial x_i} - \frac{\dfrac{\partial^2 \nu}{\partial x_i^2}}{\dfrac{\partial \nu}{\partial x_i}} \tag{2.10}$$

The first term in this equation is simply the numeraire risk aversion coefficient multiplied by the appropriate scale factor. The second term is the single-attribute risk aversion that is induced by nonlinearity in the numeraire function.

Multiattribute Risk Aversion Coefficient

The definition of multiattribute risk aversion was suggested by Richard (1975). It identifies a cross-dimensional aspect of risk aversion, which is implied by risk preference over a numeraire. In this section we first describe multiattribute risk aversion qualitatively; then we define a new coefficient, which provides us with a quantitative measure of it.

Multiattribute risk aversion is illustrated using the pair of lotteries in Figure 2.13. Looking only at attribute x_i, L_1 and L_2 are identical propositions: in each case the decision maker has probability one-half of receiving $\bar{x}_i + \Delta_i$ and probability one-half of receiving



Multiattribute risk aversion implies that $L_2 \succ L_1$.

Figure 2.13. A PAIR OF LOTTERIES FOR ILLUSTRATING MULTIATTRIBUTE RISK AVERSION ($\Delta_i > 0$, $\Delta_j > 0$, $\underline{x}_{ij} = \underline{\bar{x}}_{ij}$).

$\bar{x}_i - \Delta_i$. This is also true when looking only at attribute x_j. Looking at L_1 and L_2 in total, however, we see that L_1 gives the decision maker either the better of each outcome or the worse of each outcome whereas L_2 guarantees getting the better of one and the worse of the other.

Suppose that x_i is a decision maker's income this year, that x_j is his income next year, and that he may choose between L_1 and L_2. Then, for example, by choosing L_2 he could guarantee himself $22,000 in one year and $18,000 in the other, whereas L_1 would give him either $22,000 in both years or $18,000 in both years.

We expect that if this decision maker is risk averse, in some sense, he would choose the "safer" proposition L_2. This behavior captures the qualitative notion of multiattribute risk aversion.

To make this notion more precise we would like to express quantitatively the amount by which L_2 is preferred. From Equation (2.9), the approximation formula

$$\tilde{v} \approx \bar{v} - \frac{1}{2} r(\bar{v}) \overset{\vee}{v} \tag{2.11}$$

would be useful if we knew the mean $\bar{v}$ and variance $\overset{\vee}{v}$ of the lottery over the numeraire. Using a Taylor expansion, Howard (1971) shows that for any lottery $\{\underline{x}|\mathcal{S}\}$ with mean $\bar{\underline{x}}$ the moments of $v(\underline{x})$ are approximately

$$\bar{v} \approx v(\bar{\underline{x}}) + \frac{1}{2} \sum_{i,j} \left.\frac{\partial^2 v}{\partial x_i \partial x_j}\right|_{\bar{\underline{x}}} \operatorname{cov}(x_i, x_j) \tag{2.12}$$

$$\overset{\vee}{v} \approx \sum_{i,j} \left.\frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}\right|_{\bar{\underline{x}}} \operatorname{cov}(x_i, x_j) \tag{2.13}$$

Equations (2.12) and (2.13) hold exactly for quadratic and planar $v(\underline{x})$, respectively. Substituting these approximations into (2.11), we use the second order approximation for the first term and first order approximations elsewhere:

$$\tilde{v} \approx v(\bar{\underline{x}}) + \frac{1}{2} \sum_{i,j} \left. \frac{\partial^2 v}{\partial x_i \partial x_j} \right|_{\bar{\underline{x}}} \mathrm{cov}(x_i, x_j) - \frac{1}{2} r\left(v(\bar{\underline{x}})\right) \sum_{i,j} \left. \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \right|_{\bar{\underline{x}}} \mathrm{cov}(x_i, x_j)$$

Rearranging,

$$\tilde{v} \approx v(\bar{\underline{x}}) - \frac{1}{2} \sum_{i,j} \left[r_v \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} - \frac{\partial^2 v}{\partial x_i \partial x_j} \right]_{\bar{\underline{x}}} \mathrm{cov}(x_i, x_j)$$

It will be useful to name the quantity in brackets.

<u>Definition 2.12</u>. Multiattribute risk aversion coefficient $r_{ij}(\underline{x})$.

$$r_{ij}(\underline{x}) = r_v \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} - \frac{\partial^2 v}{\partial x_i \partial x_j}$$

Using this definition

$$\tilde{v} \approx v(\bar{\underline{x}}) - \frac{1}{2} \sum_{i,j} r_{ij}(\bar{\underline{x}}) \ \mathrm{cov}(x_i, x_j) \tag{2.14}$$

The multiattribute risk aversion coefficient consists of two terms, which can be interpreted separately. The first term $r_v(\partial v/\partial x_i)(\partial v/\partial x_j)$ when multiplied by $\mathrm{cov}(x_i, x_j)$ is the approximate risk premium due to the variance of the numeraire: when r_v is positive, increasing covariance $\mathrm{cov}(x_i, x_j)$ decreases the numeraire certain equivalent $\tilde{v}$. The second term $-\partial^2 v/\partial x_i \partial x_j$ when multiplied by $\mathrm{cov}(x_i, x_j)$ is simply the

correction factor in the approximation of the numeraire mean that arises from nonlinearity in the numeraire function $\nu(\underline{x})$; this is evident from Equation (2.12).

Summing these effects over each pair of attributes yields Equation (2.14). This equation allows us to approximate the numeraire certain equivalent of any multiattribute lottery using just the vector of means and the variance-covariance matrix of the outcome vector $\underline{x}$.

When the outcomes have only one dimension, the obvious numeraire is the outcome itself $(\nu(x) = x)$. In this case the multiattribute risk aversion coefficient reduces to Pratt's risk aversion coefficient $(r_{ij}(\underline{x}) = r_\nu(x) = r(x))$, and Equation (2.14) reduces to Equation (2.9).

The well-known result that Equation (2.9) is exact for a normal lottery and an exponential utility function can now be generalized.

<u>Theorem 2.2</u>. Equation (2.14) is exact if all of the following are satisfied:

(i) $u(\underline{x})$ is a product-of-exponentials:

$$u(\underline{x}) = -a\, e^{-\gamma_1 x_1 - \dots - \gamma_n x_n} \qquad \text{where } a\gamma_i > 0$$

(ii) $\{\underline{x}|\mathcal{S}\}$ is a multivariate normal lottery

(iii) $\nu(\underline{x})$ is either a single-attribute or a diagonal numeraire.

This is proven in Appendix B and illustrated by the example in Section 2.6.

The multiattribute risk aversion coefficients have other important interpretations. First, it is not surprising that we can combine Equation (2.10) and Definition 2.12 to yield

$$r_{ii}(\underline{x}) = r_i(\underline{x}) \frac{\partial v}{\partial x_i} \tag{2.15}$$

This means that the coefficient $r_{ii}(\underline{x})$ which multiplies the variance of x_i in the approximation formula is just the single-attribute risk aversion coefficient for x_i multiplied by an appropriate scale factor.

For $i \neq j$ the coefficients $r_{ij}(\underline{x})$ are useful for interpreting and defining multiattribute risk aversion. For L_1 and L_2, refer to Figure 2.13.

<u>Theorem 2.3</u>. If for $\bar{x}_i - \Delta_i \leq x_i \leq \bar{x}_i + \Delta_i$ and $\bar{x}_j - \Delta_j \leq x_j \leq \bar{x}_j + \Delta_j$

$$r_{ij}(\underline{x}) > 0 \qquad \text{then} \quad L_2 \succ L_1$$

$$r_{ij}(\underline{x}) = 0 \qquad \text{then} \quad L_2 \sim L_1$$

$$r_{ij}(\underline{x}) < 0 \qquad \text{then} \quad L_1 \succ L_2$$

where $i \neq j$ and $\underline{x}_{ij} = \bar{\underline{x}}_{ij}$.

Adapting a method used by Richard (197ᵉ), this result is easily verified.

<u>Proof</u>. For utility function $u(v(\underline{x}))$ it can be shown directly that

$$r_{ij}(\underline{x}) = r_v \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} - \frac{\partial^2 v}{\partial x_i \partial x_j} = - \frac{\dfrac{\partial^2 u}{\partial x_i \partial x_j}}{\dfrac{du}{dv}}$$

where $du/dv > 0$ by assumption. If $r_{ij}(\underline{x}) > 0$ within the region specified, then

$$-\frac{\dfrac{\partial^2 u}{\partial x_i \partial x_j}}{\dfrac{du}{d\nu}} > 0$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} < 0$$

Therefore, considering attributes other than x_i and x_j fixed,

$$\begin{aligned}\frac{1}{2}\int_{\bar{x}_i-\Delta_i}^{\bar{x}_i+\Delta_i}\int_{\bar{x}_j-\Delta_j}^{\bar{x}_j+\Delta_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\,dx_j dx_i = {} & \frac{1}{2}\,u(\bar{x}_i + \Delta_i, \bar{x}_j + \Delta_j) \\ & + \frac{1}{2}\,u(\bar{x}_i - \Delta_i, \bar{x}_j - \Delta_j) \\ & - \frac{1}{2}\,u(\bar{x}_i + \Delta_i, \bar{x}_j - \Delta_j) \\ & - \frac{1}{2}\,u(\bar{x}_i - \Delta_i, \bar{x}_j + \Delta_j) < 0\end{aligned}$$

where we recognize this quantity as the expected utility of L_1 minus the expected utility of L_2. Therefore, L_2 is preferred. The proofs for $r_{ij}(\underline{x}) = 0$ and $r_{ij}(\underline{x}) < 0$ follow in the same manner. Q.E.D.

Theorem 2.3 says that if $r_{ij}(\underline{x}) > 0$ over the rectangle spanned by the prizes of the lotteries, then the "safer" proposition L_2 is preferred. So, locally, we shall say that if $r_{ij}(\underline{x}) > 0$ the decision maker is <u>multiattribute risk averse</u>, if $r_{ij}(\underline{x}) = 0$ he is <u>multiattribute risk indifferent</u>, and if $r_{ij}(\underline{x}) < 0$ he is <u>multiattribute risk preferring</u>.

Let $\tilde{\nu}_1$ and $\tilde{\nu}_2$ be the respective certain equivalents for L_1 and and L_2. One quantitative measure of multiattribute risk aversion is the difference $\tilde{\nu}_2 - \tilde{\nu}_1$. Using Equation (2.14)

$$\tilde{v}_2 \approx v(\bar{\underline{x}}) - \frac{1}{2}\left[r_{ii}(\bar{\underline{x}})\,\Delta_i^2 + r_{jj}(\bar{\underline{x}})\,\Delta_j^2 - 2r_{ij}(\bar{\underline{x}})\,\Delta_i\Delta_j\right]$$

$$\tilde{v}_1 \approx v(\bar{\underline{x}}) - \frac{1}{2}\left[r_{ii}(\bar{\underline{x}})\,\Delta_i^2 + r_{jj}(\bar{\underline{x}})\,\Delta_j^2 + 2r_{ij}(\bar{\underline{x}})\,\Delta_i\Delta_j\right]$$

Subtracting,

$$\tilde{v}_2 - \tilde{v}_1 \approx 2r_{ij}(\bar{\underline{x}})\,\Delta_i\Delta_j \qquad (2.16)$$

This equation gives us a simple interpretation for $r_{ij}(\underline{x})$ as a measure of multiattribute risk aversion.

2.4 Summary of Preference Measures and Identities

Three ordinal preference measures and their interpretations, discussed in Section 2.2, are summarized in Figure 2.14. These preference measures apply to any value function, numeraire function, or multiattribute utility function that has two continuous derivatives.

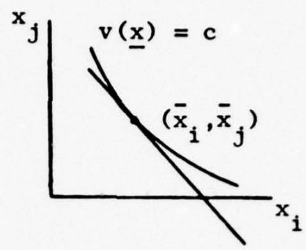
Three measures of risk preference and their interpretations, discussed in Section 2.3, are summarized in Figure 2.15. These preference measures apply to any multiattribute utility function that has two continuous derivatives.

Figure 2.16 lists ten identities relating the preference measures. Identities (1) and (2) follow directly from the definition of the marginal rate of substitution. Identities (3) and (4) are derived in Appendix C. (3) is interpreted after defining the substitution aversion coefficient in Section 2.2, and (4) says that corresponding substitution aversion coefficients are simply related by the scale factor λ.

$v(\underline{x})$ is an ordinal preference function

Marginal rate of substitution $\lambda_{ij}(\underline{x})$

$$\lambda_{ij}(\underline{x}) = \frac{\frac{\partial v}{\partial x_i}}{\frac{\partial v}{\partial x_j}}$$



slope $= -\lambda_{ij}(\bar{\underline{x}})$

$\lambda_{ij}(\bar{\underline{x}})$ is the answer to:

"Starting at $\bar{\underline{x}}$, how much x_j are you just willing to give up in order to get an additional unit of x_i?"

Marginal value reduction coefficient $z_{ij}(\underline{x})$

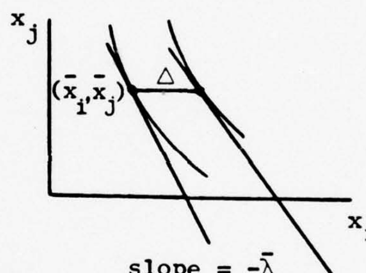
$$z_{ij}(\underline{x}) = -\frac{\frac{\partial \lambda}{\partial x_i}}{\lambda}$$

Specify Δ such that λ declines by the factor $1/e$. Then

$$\zeta_{ij}(\bar{\underline{x}}) \approx \Delta$$

where

$$\zeta_{ij}(\underline{x}) = \frac{1}{z_{ij}(\underline{x})}$$ marginal value preservation tolerance



slope $= -\bar{\lambda}$

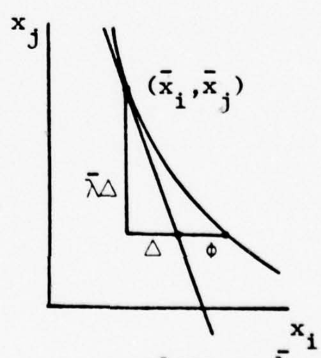
slope $= -\frac{\bar{\lambda}}{e}$

Substitution aversion coefficient $s_{ij}(\underline{x})$

$$s_{ij}(\underline{x}) = -\frac{\frac{d\lambda}{dx_i}}{\lambda}$$

$$\phi \approx \frac{1}{2}\, s_{ij}(\bar{\underline{x}})\, \Delta^2$$

$$\sigma_{ij}(\underline{x}) = \frac{1}{s_{ij}(\underline{x})}$$ substitution tolerance



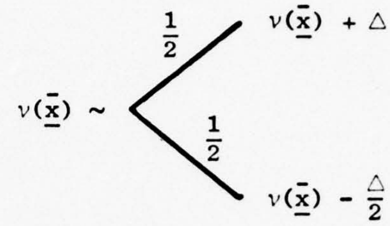
slope $= -\bar{\lambda}$

Figure 2.14. SUMMARY OF ORDINAL PREFERENCE MEASURES.

$v(\underline{x})$ is an ordinal preference function.
$\nu(\bar{\underline{x}})$ is a numeraire function corresponding to v.
$u(\nu(\underline{x}))$ is a multiattribute utility function.

Numeraire risk aversion coefficient $r_\nu(\underline{x})$

$$r_\nu(\underline{x}) = - \left. \frac{\frac{d^2u}{d\nu^2}}{\frac{du}{d\nu}} \right|_{\nu(\underline{x})}$$

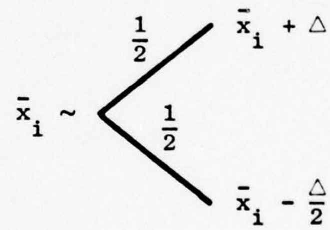


Specify Δ such that indifference holds. Then

$$r_\nu(\bar{\underline{x}}) \approx \frac{1}{\Delta}$$

Single-attribute risk aversion coefficient $r_i(\underline{x})$

$$r_i(\underline{x}) = - \left. \frac{\frac{\partial^2 u}{\partial x_i^2}}{\frac{\partial u}{\partial x_i}} \right|_{\underline{x}}$$

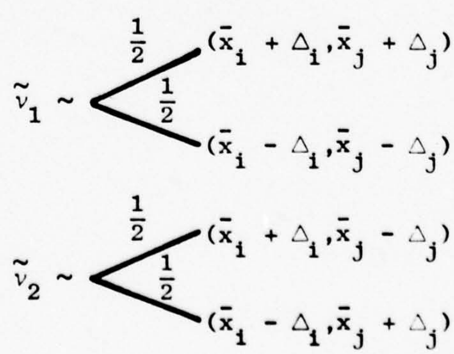


Specify Δ such that indifference holds. Then

$$r_i(\bar{\underline{x}}) \approx \frac{1}{\Delta}$$

Multiattribute risk aversion coefficient $r_{ij}(\underline{x})$

$$r_{ij}(\underline{x}) = \left[r_\nu \frac{\partial \nu}{\partial x_i} \frac{\partial \nu}{\partial x_j} - \frac{\partial^2 \nu}{\partial x_i \partial x_j} \right]_{\underline{x}}$$



Specify $\tilde{\nu}_1$ and $\tilde{\nu}_2$ such that indifference holds. Then

$$\tilde{\nu}_2 - \tilde{\nu}_1 \approx 2 r_{ij}(\bar{\underline{x}}) \Delta_i \Delta_j$$

$\Delta_i \geq 0$ and $\Delta_j \geq 0$

Figure 2.15. SUMMARY OF RISK PREFERENCE MEASURES.

Ordinal Preference Identities

(1) $$\lambda_{ij}(\underline{x}) = \lambda_{ik}(\underline{x})\ \lambda_{kj}(\underline{x})$$

(2) $$\lambda_{ij}(\underline{x}) = \frac{1}{\lambda_{ji}(\underline{x})}$$

(3) $$s_{ij}(\underline{x}) = z_{ij}(\underline{x}) + \lambda z_{ji}(\underline{x})$$

(4) $$s_{ij}(\underline{x}) = \lambda s_{ji}(\underline{x})$$

Risk Preference Identities

(5) $$r_{ij}(\underline{x}) = r_\nu \frac{\partial \nu}{\partial x_i} \frac{\partial \nu}{\partial x_j} - \frac{\partial^2 \nu}{\partial x_i \partial x_j} = - \frac{\dfrac{\partial^2 u}{\partial x_i \partial x_j}}{\dfrac{du}{d\nu}}$$

(6) $$r_{ij}(\underline{x}) = r_{ji}(\underline{x})$$

(7) $$r_{ij}(\underline{x}) = \left[r_i(\underline{x}) - z_{ij}(\underline{x}) \right] \frac{\partial \nu}{\partial x_j}$$

(8) $$r_{ii}(\underline{x}) = r_i(\underline{x}) \frac{\partial \nu}{\partial x_i}$$

(9) $$r_i(\underline{x}) - z_{ij}(\underline{x}) = \lambda \left[r_j(\underline{x}) - z_{ji}(\underline{x}) \right]$$

(10) $$r_i(\underline{x}) = r_\nu \frac{\partial \nu}{\partial x_i} - \frac{\dfrac{\partial^2 \nu}{\partial x_i^2}}{\dfrac{\partial \nu}{\partial x_i}}$$

Figure 2.16. PREFERENCE MEASURE IDENTITIES ($\lambda = \lambda_{ij}(\underline{x})$).

Identity (5) is true by definition. The second expression for $r_{ij}(\underline{x})$ is useful for determining whether or not a utility function is multiattribute risk averse when it is in the form $u(\underline{x})$; in order to measure multiattribute risk aversion quantitatively, however, it is necessary to define a numeraire. (6) follows immediately from (5).

Identities (7), (8), and (9) are also derived in Appendix C. (7) suggests that for a decision maker to be multiattribute risk averse his single-attribute risk aversion coefficient must be greater than his corresponding marginal value reduction coefficient. (9) shows that corresponding differences between single-attribute risk aversion and marginal value reduction coefficients are related by the scale factor λ. Identities (8) and (10) are interpreted as Equations (2.15) and (2.10), respectively.

2.5 Deriving Preference Measures for a Particular Preference Function

In this section we show how the preference measures for any particular preference function can be derived in an efficient way.

The preference attitude in Table 2.1 is characterized by a Cobb-Douglas (hyperbolic) ordinal preference function and a product-of-powers multiattribute utility function. The independence conditions cited at the bottom of the figure are explained in Chapter 3.

In deriving the preference measures for this preference attitude, we illustrate the procedure that is generally the most convenient.

$$v(\underline{x}) = \sum_{i=1}^{n} a_i \ln x_i$$

Table 2.1

PREFERENCE MEASURES FOR A PRODUCT-OF-POWERS UTILITY FUNCTION

Value Function	Single-Attribute Numeraire	Utility over Numeraire	Multiattribute Utility Function
$v(\underline{x}) = \sum_{i=1}^{n} a_i \ln x_i$ $a_i > 0, \quad x_i > 0$	$\nu(\underline{x}) = x_1 \left(\frac{x_2}{\bar{x}_2}\right)^{\frac{a_2}{a_1}} \cdots \left(\frac{x_n}{\bar{x}_n}\right)^{\frac{a_n}{a_1}}$	$u(\nu) = -a\nu^{-\alpha}$ $\alpha \neq 0$	$u(\underline{x}) = -a x_1^{-\alpha_1} \cdots x_n^{-\alpha_n}$ $\alpha_i = \alpha \frac{a_i}{a_1}$
$\lambda_{ij}(\underline{x}) = \frac{a_i}{a_j} \frac{x_j}{x_i}$		$r_\nu(\underline{x}) = \frac{1 + \alpha}{\nu(\underline{x})}$	
$z_{ij}(\underline{x}) = \frac{1}{x_i}$		$r_i(\underline{x}) = \frac{1 + \alpha_i}{x_i}$	
$s_{ij}(\underline{x}) = \frac{1}{x_i}\left(1 + \frac{a_i}{a_j}\right)$		$r_{ij}(\underline{x}) = \frac{\nu(\underline{x})}{\alpha} \frac{\alpha_i}{x_i} \frac{\alpha_j}{x_j}$	
Deterministic Additive Independence		Mutual Utility Independence Product-of-Powers Delta Property	

$$\frac{\partial v}{\partial x_i} = \frac{a_i}{x_i}$$

$$\frac{\partial^2 v}{\partial x_i \partial x_j} = \begin{cases} -\dfrac{a_i}{x_i^2} & i = j \\[2ex] 0 & i \neq j \end{cases}$$

Now we form the marginal rate of substitution from its definition and the marginal value reduction coefficient from Equation (2.1):

$$\lambda = \frac{\dfrac{\partial v}{\partial x_i}}{\dfrac{\partial v}{\partial x_j}}$$

$$\lambda = \frac{a_i}{a_j} \frac{x_j}{x_i}$$

$$z_{ij}(\underline{x}) = - \frac{\dfrac{\partial^2 v}{\partial x_i^2}}{\dfrac{\partial v}{\partial x_i}} + \frac{\dfrac{\partial^2 v}{\partial x_i \partial x_j}}{\dfrac{\partial v}{\partial x_j}}$$

$$= \frac{1}{x_i} \qquad \text{for} \quad i \neq j$$

Then we find the substitution aversion coefficient using the identity

$$s_{ij}(\underline{x}) = z_{ij}(\underline{x}) + \lambda z_{ji}(\underline{x})$$

$$= \frac{1}{x_i} + \lambda \frac{1}{x_j}$$

$$= \frac{1}{x_i}\left(1 + \frac{a_i}{a_j}\right) \qquad \text{for} \quad i \neq j$$

By definition, the numeraire risk aversion coefficient and the single-attribute risk aversion coefficient are

$$r_\nu(\underline{x}) = -\frac{\dfrac{d^2u}{d\nu^2}}{\dfrac{du}{d\nu}}$$

$$= \frac{1+\alpha}{\nu(\underline{x})}$$

and

$$r_i(\underline{x}) = -\frac{\dfrac{\partial^2 u}{\partial x_i^2}}{\dfrac{\partial u}{\partial x_i}}$$

$$= \frac{1+\alpha_i}{x_i}$$

An alternative method for finding the single-attribute risk aversion coefficient will be easiest in many cases: differentiate the numeraire twice and use Equation (2.10):

$$r_i(\underline{x}) = r_\nu \frac{\partial \nu}{\partial x_i} - \frac{\dfrac{\partial^2 \nu}{\partial x_i^2}}{\dfrac{\partial \nu}{\partial x_i}}$$

To form the multiattribute risk aversion coefficient, we use the first derivative of the numeraire

$$\frac{\partial \nu}{\partial x_j} = \frac{a_j}{a_1}\frac{\nu(\underline{x})}{x_j}$$

and the identity

$$r_{ij}(\underline{x}) = \left[r_i(\underline{x}) - z_{ij}(\underline{x})\right] \frac{\partial \nu}{\partial x_j}$$

$$= \left[\frac{1 + \alpha_i}{x_i} - \frac{1}{x_i}\right] \frac{a_j}{a_1} \frac{\nu(\underline{x})}{x_j}$$

$$= \frac{\nu(\underline{x})}{\alpha} \frac{\alpha_i}{x_i} \frac{\alpha_j}{x_j}$$

In interpreting this preference attitude, note that $s_{ij}(\underline{x})$ is a decreasing function of wealth in x_i but does not depend on wealth in any other attribute. Also note that, for $\alpha > 0$, $r_{ij}(\underline{x})$ is positive but increases as x_k increases for $k \neq i,j$. Interpretations such as these should be carefully scrutinized before using a preference function for practical modeling.

Appendix D lists the preference measures for a variety of particular preference functions.

2.6 Example: Risk Preference over Discounted Cash Flows

Applying the methods of this chapter, we shall now construct and interpret a simple preference model for cash flows.

Let x_i be the decision maker's income in period i. Then

$$\underline{x} = (x_o, \ldots, x_n)'$$

represents a cash flow. Let us assume that at any time the decision maker can borrow or invest any amount at the interest rate $\hat{i}$. Defining the discount rate β

$$\beta = \frac{1}{1 + \hat{i}}$$

enables us to write the net present value v of the cash flow $\underline{x}$ as

$$v(\underline{x}) = x_o + \beta x_1 + \ldots + \beta^n x_n$$

$$= \underline{\beta}\,\underline{x} \qquad \text{where } \underline{\beta} = (1, \beta, \ldots, \beta^n)$$

Defined in this way, the net present value is a natural single-attribute numeraire: $v(\underline{x})$ is the present amount that is equivalent to the cash flow $\underline{x}$. Let us analyze this ordinal preference attitude.

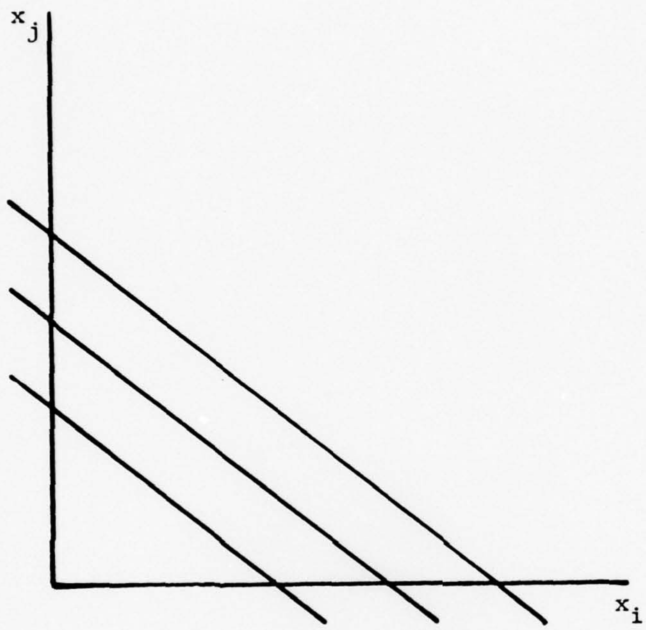
$$\lambda_{ij}(\underline{x}) = \frac{\frac{\partial v}{\partial x_i}}{\frac{\partial v}{\partial x_j}}$$

$$= \beta^{i-j}$$

Thus, the decision maker's indifference curves are parallel straight lines as shown in Figure 2.17. This also implies an infinite tolerance for the preservation of marginal value and an infinite tolerance for substitution at the marginal rate β^{i-j}. While these conditions might not be appropriate over a wide range of cash flows, they are often satisfactory for particular problems.

Over a limited range, the decision maker's risk preference over net present values can usually be modeled as exponential:

$$u(v) = -a\, e^{-\gamma v}$$



slope $= -\beta^{i-j}$

$\lambda_{ij}(\underline{x}) = \beta^{i-j}$

$z_{ij}(\underline{x}) = 0$

$s_{ij}(\underline{x}) = 0$

Figure 2.17. DISCOUNTED CASH FLOW INDIFFERENCE CURVES.

$$u\left(\nu(\underline{x})\right) = -a\ e^{-\gamma x_0 - \gamma\beta x_1 - \ldots - \gamma\beta^n x_n}$$

$$= -a\ e^{-\gamma \underline{\beta}\ \underline{x}}$$

For this product-of-exponentials risk attitude

$$r_\nu(\underline{x}) = -\frac{\dfrac{d^2u}{d\nu^2}}{\dfrac{du}{d\nu}}$$

$$= \gamma$$

$$r_i(\underline{x}) = -\frac{\dfrac{\partial^2 u}{\partial x_i^2}}{\dfrac{\partial u}{\partial x_i}}$$

$$= \gamma\beta^i$$

$$r_{ij}(\underline{x}) = r_\nu \frac{\partial \nu}{\partial x_i}\frac{\partial \nu}{\partial x_j} - \frac{\partial^2 \nu}{\partial x_i \partial x_j}$$

$$= \gamma\beta^{i+j}$$

Then, if $\gamma > 0$, the decision maker is risk averse toward lotteries over net present value, risk averse toward lotteries over each period's income by itself, and multiattribute risk averse over any pair of incomes. The two latter coefficients have the property that the decision maker is less risk averse toward lotteries over outcomes that are further in the future.

Finally, suppose that the decision maker's uncertainty over his future income stream can be modeled as a multivariate normal lottery with mean

$$\underline{\mu} = (\mu_o, \ldots, \mu_n)'$$

and variance-covariance matrix

$$A = \left[a_{ij}\right]$$

where $a_{ij} = \text{cov}(x_i, x_j)$. We can use Equation (2.14) to approximate the net present value certain equivalent for this lottery.

$$\tilde{\nu} \approx \nu(\underline{\mu}) - \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} r_{ij}(\underline{\mu})\, a_{ij}$$

$$\tilde{\nu} = \underline{\beta}\,\underline{\mu} - \frac{1}{2}\,\gamma\underline{\beta}A\underline{\beta}'$$

In this case the approximation is exact, according to Theorem 2.2, because the decision maker's utility function is a product-of-exponentials, his uncertainty is multivariate normal, and the net present value is a single-attribute numeraire.

Chapter 3

INDEPENDENCE CONDITIONS AND DELTA PROPERTIES FOR LIMITING FUNCTIONAL FORM

The purpose of this chapter is to identify some independence conditions and delta properties that are simple global interpretations of multiattribute preferences. These are useful in Chapter 4 for limiting the functional form of a preference attitude.

This chapter is not intended to summarize the large literature on independence conditions for multiattribute preference functions. Rather, it presents only the most fundamental independence results and one or two salient interpretations of each. Also, whenever possible, new interpretations are provided in terms of the local preference measures of Chapter 2.

Risk additive independence and deterministic additive independence are my own terminology. Their purpose is to make a clear distinction between additive utility functions and additive value functions.

3.1 Risk Additive Independence

Risk additive independence is a strong condition that is useful for interpreting additive utility functions. The following theorem and the definition that precedes it are due conceptually to Fishburn (1965).

Definition 3.1. Risk additive independence. Attributes $x_1,\ldots,x_n$ are risk additive independent if preferences over lotteries on $(x_1,\ldots,x_n)$ depend only on the marginal probability distributions $\{x_1|\&\},\ldots,\{x_n|\&\}$ and not on the joint probability distribution $\{x_1,\ldots,x_n|\&\}$.

A pairwise interpretation of risk additive independence, which is equivalent to this definition, is illustrated in Figure 3.1: let $\bar{\underline{x}}$ be any nominal wealth level; then attributes $x_1,\ldots,x_n$ are risk additive independent if and only if the decision maker is indifferent between the two lotteries shown for all i, j, x_i, and x_j, where $\underline{x}_{-ij} = \bar{\underline{x}}_{-ij}$. Figure 2.13 provides an example of this behavior if $L_1 \sim L_2$.

The following theorem is the fundamental result of additive utility theory.

<u>Theorem 3.1.</u> (Fishburn). A multiattribute utility function is additive

$$u(\underline{x}) = u_1(x_1) + \ldots + u_n(x_n)$$

if and only if attributes $x_1,\ldots,x_n$ are risk additive independent.

<u>Proof</u>. To see that an additive utility function implies risk additive independence, observe that the expected utility of any lottery depends only on its marginal distributions:

$$\int_{\underline{x}} \{\underline{x} \mid \delta\} \left[u_1(x_1) + \ldots + u_n(x_n)\right]$$

$$= \int_{\underline{x}} \{\underline{x} \mid \delta\}\, u_1(x_1) + \ldots + \int_{\underline{x}} \{\underline{x} \mid \delta\}\, u_n(x_n)$$

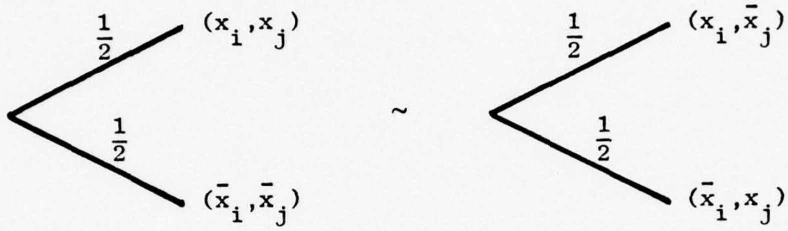
$$= \int_{x_1} \{x_1 \mid \delta\}\, u_1(x_1) + \ldots + \int_{x_n} \{x_n \mid \delta\}\, u_n(x_n)$$

To show that risk additive independence implies an additive utility function, Keeney and Raiffa (1975) use the following observation: the

$x_1, \ldots, x_n$

are risk additive independent

if



for all i, j, x_i, and x_j, where $\underline{x}_{ij} = \underline{\bar{x}}_{ij}$.

Figure 3.1. AN INTERPRETATION OF RISK ADDITIVE INDEPENDENCE ($\underline{\bar{x}}$ IS ANY NOMINAL WEALTH LEVEL).

two lotteries in Figure 3.1 have identical marginal distributions and, therefore, their expected utilities must be equal. For $n = 2$,

$$\frac{1}{2}u(x_1,x_2) + \frac{1}{2}u(\bar{x}_1,\bar{x}_2) = \frac{1}{2}u(x_1,\bar{x}_2) + \frac{1}{2}u(\bar{x}_1,x_2)$$

Defining

$$u_1(x_1) = u(x_1,\bar{x}_2)$$

and

$$u_2(x_2) = u(\bar{x}_1,x_2)$$

and scaling $u(\underline{x})$ so that $u(\bar{x}_1,\bar{x}_2) = 0$ yields

$$u(x_1,x_2) = u_1(x_1) + u_2(x_2)$$

Assume then that $u(\underline{x})$ is additive for $n-1$ attributes. For n attributes

$$u(x_1,\ldots, x_n) = f_1(x_1,x_n) + \ldots + f_{n-1}(x_{n-1},x_n)$$

Applying the lotteries in Figure 3.1 as above for each $i = 1,\ldots,n-1$, we can write

$$f_i(x_i,x_n) = u_i(x_i) + g_i(x_n)$$

from which the desired result immediately follows. Q.E.D.

Definition 3.1 and Theorem 3.1 apply in the general case where each attribute is itself a vector of attributes. For example, if y_1 and y_2 are risk additive independent where $y_1 = (x_1,x_2)$ and $y_2 = (x_3,x_4)$, then Theorem 3.1 assures us that

$$u(\underline{x}) = u_1(x_1,x_2) + u_2(x_3,x_4)$$

When each attribute is a scalar, risk additive independence has important interpretations in terms of the decision maker's local preference attitude. First, it is easy to show that risk additive independence is equivalent to multiattribute risk indifference.

<u>Theorem 3.2</u>. A multiattribute utility function is additive

$$u(\underline{x}) = u_1(x_1) + \ldots + u_n(x_n)$$

if and only if for all $i \neq j$

$$r_{ij}(\underline{x}) = 0 \qquad \text{for all } \underline{x}$$

<u>Proof</u>. If $u(\underline{x})$ is additive, $r_{ij}(\underline{x}) = 0$ from Identity 5 in Figure 2.16. If $r_{ij}(\underline{x}) = 0$ for all $\underline{x}$, then, by Theorem 2.3, the decision maker is always indifferent between L_1 and L_2 in Figure 2.13. This condition is equivalent to the interpretation of risk additive independence in Figure 3.1 from which we have shown that $u(\underline{x})$ is additive.

Q.E.D.

Therefore, the decision maker's utility function is additive if and only if he is always indifferent between L_1 and L_2 in Figure 2.13. This interpretation of additive utility is often the easiest one to use in practical application.

A second interpretation of risk additive independence for scalar attributes arises from recognizing that an additive utility function also represents an additive ordinal preference attitude.

Theorem 3.3. If $u(\underline{x}) = u_1(x_1) + \ldots + u_n(x_n)$, then $r_i(\underline{x}) = z_{ij}(\underline{x})$ for $j \neq i$.

Proof. If $u(\underline{x}) = u_1(x_1) + \ldots + u_n(x_n)$,

$$\lambda = \frac{\dfrac{du_i}{dx_i}}{\dfrac{du_j}{dx_j}}$$

$$\frac{\partial\lambda}{\partial x_i} = \frac{\dfrac{d^2u_i}{dx_i^2}}{\dfrac{du_j}{dx_j}} = \lambda \frac{\dfrac{d^2u_i}{dx_i^2}}{\dfrac{du_i}{dx_i}} \qquad \text{for } j \neq i$$

Then, by definition

$$z_{ij}(\underline{x}) = - \frac{\dfrac{\partial\lambda}{\partial x_i}}{\lambda} = - \frac{\dfrac{d^2u_i}{dx_i^2}}{\dfrac{du_i}{dx_i}} = - \frac{\dfrac{\partial^2u}{\partial x_i^2}}{\dfrac{\partial u}{\partial x_i}} = r_i(\underline{x})$$

Q.E.D.

This theorem says that a decision maker with an additive utility function must be willing to interpret his marginal value reduction coefficient $z_{ij}(\underline{x})$ and single-attribute risk aversion coefficient $r_i(\underline{x})$ as identical quantities. Note that both coefficients in this case depend only on attribute x_i.

3.2 Delta Properties for Additive Utility Functions

If a decision maker's preferences satisfy risk additive independence, then we can proceed to ask some additional lottery questions to

see whether we can further restrict functional form. The following theorem, which is interpreted in Figures 3.2 and 3.3, gives behavioral conditions for $u_i(x_i)$ to be exponential, linear, logarithmic, or a power form.

Theorem 3.4. Delta properties for additive utility functions. Assume that $u(\underline{x})$ is additive

$$u(\underline{x}) = u_1(x_1) + \ldots + u_n(x_n)$$

and let $\underline{x}_{\bar{i}} = \bar{\underline{x}}_{\bar{i}}$. Then $u_i(x_i)$ can be written

$$\text{(i)} \qquad u_i(x_i) = \begin{cases} -a_i\, e^{-\gamma_i x_i} & \text{where } \gamma_i \neq 0 \\ a_i x_i & \text{where } \gamma_i = 0 \end{cases}$$

if and only if for any lottery $\{x_i|\mathcal{S}\}$

$$u_i(\tilde{x}_i) = \int_{x_i} \{x_i|\mathcal{S}\}\, u_i(x_i)$$

implies

$$u_i(\tilde{x}_i + \Delta) = \int_{x_i} \{x_i|\mathcal{S}\}\, u_i(x_i + \Delta) \qquad \text{for any } \Delta$$

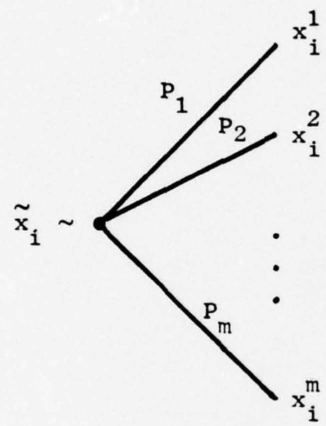
or

$$\text{(ii)} \qquad u_i(x_i) = \begin{cases} -a_i x_i^{-\alpha_i} & \text{where } x_i > 0,\ \alpha_i \neq 0 \\ a_i \ln x_i & \text{where } x_i > 0,\ \alpha_i = 0 \end{cases}$$

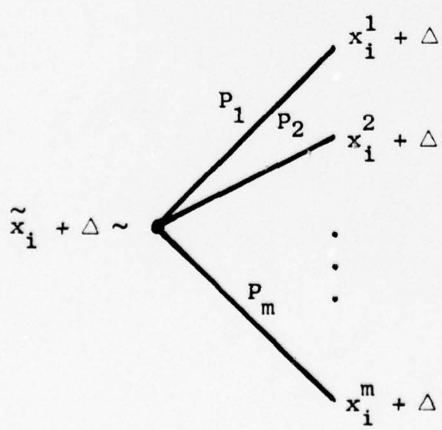
Assume $u(\underline{x}) = u_1(x_1) + \ldots + u_n(x_n)$. Then

$$u_i(x_i) = \begin{cases} -a_i e^{-\gamma_i x_i} \\ \text{or} \\ a_i x_i \end{cases}$$

if and only if



implies



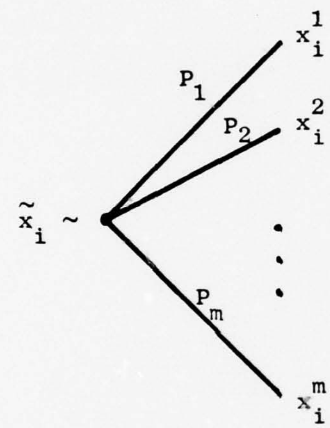
for any Δ

Figure 3.2. EXPONENTIAL DELTA PROPERTY FOR ADDITIVE UTILITY FUNCTIONS ($\underline{x}_i = \bar{\underline{x}}_i$; FOR x_i^k, k IS A SUPERSCRIPT).

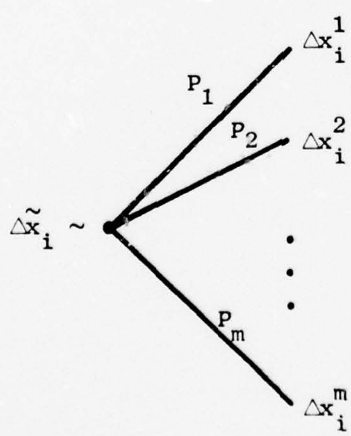
Assume $u(\underline{x}) = u_1(x_1) + \ldots + u_n(x_n)$. Then

$$u_i(x_i) = \begin{cases} -a_i x_i^{-\alpha_i} \\ \quad \text{or} \\ a_i \ln x_i \end{cases}$$

if and only if



implies



for any $\Delta > 0$

Figure 3.3. LOGARITHMIC DELTA PROPERTY FOR ADDITIVE UTILITY FUNCTIONS ($x_i > 0$; $\underline{x}_i = \bar{x}_i$; FOR x_i^k, k IS A SUPERSCRIPT).

if and only if for any lottery $\{x_i | \mathcal{S}\}$ where $x_i > 0$

$$u_i(\tilde{x}_i) = \int_{x_i} \{x_i | \mathcal{S}\}\, u_i(x_i)$$

implies

$$u_i(\Delta\tilde{x}_i) = \int_{x_i} \{x_i | \mathcal{S}\}\, u_i(\Delta x_i) \qquad \text{for any } \Delta > 0$$

This theorem is proven in Appendix E. Part (i) says that if increasing all of the prizes of a lottery over x_i by Δ increases the lottery's certain equivalent by Δ, then $u_i(x_i)$ must be either linear or exponential. Whether the appropriate form is linear or exponential depends on the decision maker's risk aversion coefficient $r_i(\underline{x}) = \gamma_i$ as indicated.

Part (ii) assumes that $x_i > 0$ and says that if multiplying all of the prizes of a lottery by a positive constant Δ multiplies the lottery's certain equivalent by Δ, then $u_i(x_i)$ must be either a logarithmic or a power form. Again the appropriate form depends on the decision maker's risk aversion coefficient $r_i(\underline{x}) = (1 + \alpha_i)/x_i$.

Note that this theorem applies to single-attribute utility functions as a special case. Also, an expected value decision maker $(u_i(x_i) = a_i x_i)$ must satisfy both parts (i) and (ii), regardless of the sign of x_i.

3.3 Utility Independence

Utility independence is useful for interpreting additive, multiplicative, and multilinear utility functions. We shall discuss only the additive and multiplicative forms. Most of the results in this section

are due to Keeney (1971, 1972, 1974), and a broader discussion of utility independence, along with a proof of Theorem 3.5, can be found in Keeney and Raiffa (1975). Note that the results in this section are valid in the general case where each attribute x_i is itself a vector of attributes.

Definition 3.2. Utility independence. Let y be a subset of $x_1, \ldots, x_n$. Then y is utility independent of its complement y^c if for any pair of lotteries $\{y|\delta_1\}$ and $\{y|\delta_2\}$

$$\{y|\delta_1\} \succ \{y|\delta_2\} \qquad \text{for some fixed } y^c$$

implies that

$$\{y|\delta_1\} \succ \{y|\delta_2\} \qquad \text{for any } y^c$$

In words, this means that the preference ordering over all lotteries on y is independent of the fixed levels of all attributes not contained in y. An equivalent interpretation, which is illustrated in Figure 3.4, is that a certain equivalent for any lottery on y does not depend on the fixed level of y^c.

Attributes $x_1, \ldots, x_n$ are said to be mutually utility independent if every subset of $x_1, \ldots, x_n$ is utility independent of its complement.

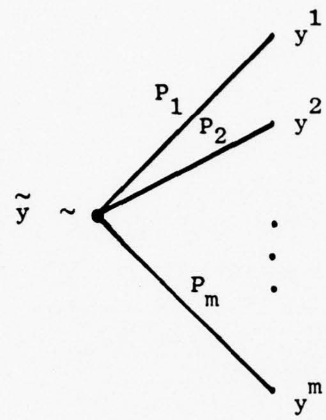
Theorem 3.5 (Keeney). A multiattribute utility function has either an additive or multiplicative form

$$u(\underline{x}) = u_1(x_1) + \ldots + u_n(x_n)$$

or

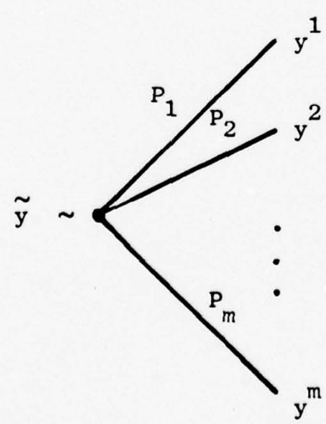
$$u(\underline{x}) = u_1(x_1)\, u_2(x_2) \ldots u_n(x_n)$$

y is utility independent of its complement y^c if



for some fixed y^c

implies



for any y^c

Figure 3.4. AN INTERPRETATION OF UTILITY INDEPENDENCE.

if and only if the attributes $x_1, \ldots, x_n$ are mutually utility independent.

When $n = 2$, mutual utility independence is easy to visualize: the certain equivalent for any lottery in one attribute must not depend on the fixed level of the other attribute, and vice versa. As n grows, however, lottery comparisons among all subsets of $x_1, \ldots, x_n$ are increasingly difficult to think about. The following theorem, also discussed by Keeney and Raiffa, interprets mutual utility independence in terms of its underlying ordinal preference attitude. This not only will lead to, in the next section, a more practical interpretation for utility independent preferences when $n \geq 3$, but also suggests a useful method for assessment.

<u>Theorem 3.6.</u>

(i) A decision maker's ordinal preference function $v(\underline{x})$ can be written in the additive form

$$v(\underline{x}) = v_1(x_1) + \ldots + v_n(x_n)$$

and

(ii) attribute x_1 is utility independent of its complement $(x_2, \ldots, x_n)$

if and only if the multiattribute utility function can be written

$$u(\underline{x}) = \begin{cases} -a\, e^{-\gamma' [v_1(x_1) + \ldots + v_n(x_n)]} & \text{when } \gamma \neq 0 \\ v_1(x_1) + \ldots + v_n(x_n) & \text{when } \gamma = 0 \end{cases}$$

Corollary. Attributes $x_1, \ldots, x_n$ are mutually utility independent if and only if (i) and (ii) hold.

Proof of Theorem 3.6. To prove the only if part, assume that x_1 is utility independent of its complement and that $v(\underline{x})$ is additive. Let $\tilde{x}_1$ be a certain equivalent for $\{x_1 | \mathcal{S}\}$ when $(x_2, \ldots, x_n)$ is fixed at some arbitrary level. Since x_1 is utility independent of its complement and $u(\underline{x})$ can be written $u(v(\underline{x}))$,

$$u\left(v_1(\tilde{x}_1) + \sum_{i=2}^{n} v_i(x_i)\right) = \int_{x_1} \{x_1 | \mathcal{S}\}\, u\left(v_1(x_1) + \sum_{i=2}^{n} v_i(x_i)\right)$$

for any $(x_2, \ldots, x_n)$. By varying $(x_2, \ldots, x_n)$ we simultaneously increase both the prizes of the lottery on v and its certain equivalent value by the same amount. Therefore, by Theorem 3.4, the utility function over v is either linear or exponential.

To prove the if part, note that $u(\underline{x})$ is either additive or multiplicative, which implies that the attributes $x_1, \ldots, x_n$ are mutually utility independent and, in particular, that x_1 is utility independent of its complement. Further, $u(\underline{x})$ represents an additive ordinal preference function because an appropriate increasing transformation of it is additive. Q.E.D.

3.4 Delta Properties for Multiplicative Utility Functions

Mutual utility independence is a useful interpretation for multiplicative utility functions in general. In this section we provide additional interpretations for two particular multiplicative forms: the product-of-exponentials and the product-of-powers.

The multiattribute certain equivalent $\underset{\sim}{\tilde{x}}$ for a lottery $\{\underline{x}|\mathcal{S}\}$ is, of course, not unique. The theorems in this section are valid for any multiattribute certain equivalent. These theorems are illustrated in Figures 3.5 and 3.6.

Theorem 3.7. Product-of-exponentials delta property. Let $\underline{x} = (x_1, \ldots, x_n)'$, and let $\tilde{\underline{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)'$, $\underline{\gamma} = (\gamma_1, \ldots, \gamma_n)$, and $\underline{\Delta} = (\Delta_1, \ldots, \Delta_n)'$ be vectors of constants. Then a risk averse $(r_i(\underline{x}) > 0$ and $r_{ij}(\underline{x}) > 0)$ decision maker has the product-of-exponentials utility function

$$u(\underline{x}) = -e^{-\underline{\gamma}\,\underline{x}} \qquad \text{where } \gamma_i > 0$$

if and only if for any lottery $\{\underline{x}|\mathcal{S}\}$

$$u(\tilde{\underline{x}}) = \int_{\underline{x}} \{\underline{x}|\mathcal{S}\}\, u(\underline{x})$$

implies that

$$u(\tilde{\underline{x}} + \underline{\Delta}) = \int_{\underline{x}} \{\underline{x}|\mathcal{S}\}\, u(\underline{x} + \underline{\Delta})$$

for any $\underline{\Delta}$.

Thus, if adding $\underline{\Delta}$ to the prizes of a lottery always increases the certain equivalent by $\underline{\Delta}$, then the decision maker's utility function must be a product-of-exponentials.

For the product-of-exponentials utility function, we can derive a closed form numeraire certain equivalent for any lottery $\{\underline{x}|\mathcal{S}\}$. By definition

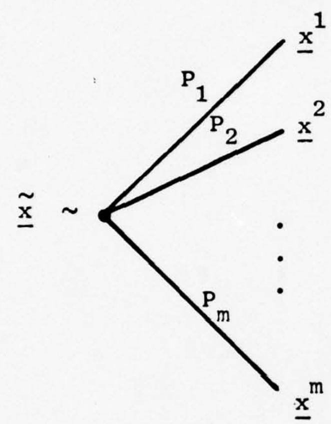
$$u(\tilde{\nu}) = -\int_{\underline{x}} \{\underline{x}|\mathcal{S}\}\, e^{-\underline{\gamma}\,\underline{x}}$$

$$= -f^{e}_{\underline{x}}(\underline{\gamma})$$

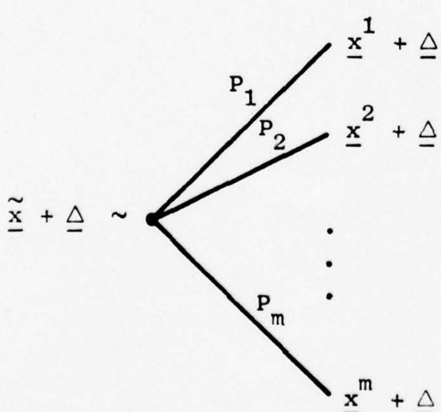
For a risk averse decision maker

$$u(\underline{x}) = -e^{-\underline{\gamma}\,\underline{x}} \qquad \text{where} \quad \gamma_i > 0$$

if and only if



implies



for any $\underline{\Delta}$

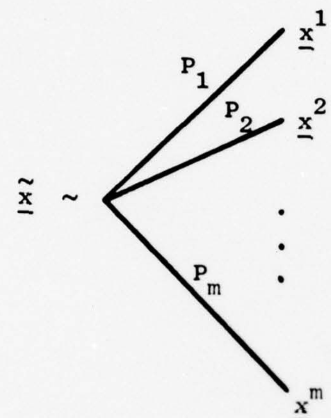
Figure 3.5. PRODUCT-OF-EXPONENTIALS DELTA PROPERTY.

For a risk averse decision maker and $x_i > 0$

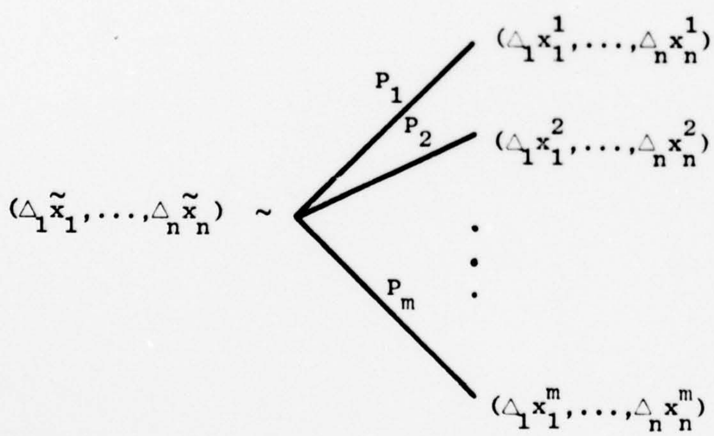
$$u(\underline{x}) = -a x_1^{-\alpha_1} x_2^{-\alpha_2} \cdots x_n^{-\alpha_n} \qquad \text{where} \quad \alpha_i > -1$$

$$a\alpha_i > 0$$

if and only if



implies



for any $\Delta_i > 0$

Figure 3.6. PRODUCT-OF-POWERS DELTA PROPERTY.

where $f^e_{\underline{x}}(\underline{s})$ is the Laplace transform of $\{\underline{x}|\mathcal{S}\}$. If $\tilde{\nu}$ is a diagonal numeraire, then

$$-e^{-\tilde{\nu}\sum_{i=1}^{n}\gamma_i} = -f^e_{\underline{x}}(\underline{\gamma})$$

$$\tilde{\nu} = -\frac{1}{\sum_{i=1}^{n}\gamma_i} \ln f^e_{\underline{x}}(\underline{\gamma}) \tag{3.2}$$

If $\tilde{\nu}$ is a single-attribute numeraire defined using $\underline{x}_1 = \hat{\underline{x}}_1$, the corresponding formula is

$$\tilde{\nu} = -\frac{1}{\gamma_1}\left[\sum_{i=2}^{n}\gamma_i\hat{x}_i + \ln f^e_{\underline{x}}(\underline{\gamma})\right] \tag{3.3}$$

Equations (3.2) and (3.3) are used in Appendix B to evaluate the numeraire certain equivalent for a multivariate normal lottery.

<u>Theorem 3.8</u>. Product-of-powers delta property. Let $x_i > 0$. Then a risk averse ($r_i(\underline{x}) > 0$ and $r_{ij}(\underline{x}) > 0$) decision maker has the product-of-powers utility function

$$u(\underline{x}) = -ax_1^{-\alpha_1} \ldots x_n^{-\alpha_n} \qquad \text{where} \quad \begin{array}{l} \alpha_i > -1 \\ a\alpha_i > 0 \end{array}$$

if and only if for any lottery $\{\underline{x}|\mathcal{S}\}$

$$u(\tilde{\underline{x}}) = \int_{\underline{x}} \{\underline{x}|\mathcal{S}\}\, u(\underline{x})$$

implies that

$$u(\Delta_1\tilde{x}_1, \ldots, \Delta_n\tilde{x}_n) = \int_{\underline{x}} \{\underline{x}|\mathcal{S}\}\, u(\Delta_1 x_1, \ldots, \Delta_n x_n)$$

for any $\Delta_i > 0$.

Thus, if multiplying each possible outcome x_i by Δ_i always multiplies the corresponding component of the certain equivalent by Δ_i, then the decision maker's utility function must be a product-of-powers.

To complete our interpretation of the product-of-exponentials and product-of-powers utility functions, the six preference measures for these two forms appear in Appendix D and Table 2.1, respectively.

Theorems 3.7 and 3.8 are proven in Appendix F. In each theorem we have assumed risk aversion for simplicity only. If the decision maker were risk preferring instead, only a couple of signs would change. Without a risk aversion assumption, we could prove a more general theorem covering products of linear, exponential, logarithmic, and power factors. These heterogeneous forms, however, imply preference reversals (see Fishburn and Keeney (1975)) and have other peculiar properties that would rarely be useful.

3.5 Deterministic Additive Independence

If a decision maker's preferences fail to satisfy either risk additive independence or mutual utility independence, then it is still possible for his underlying ordinal preference attitude to be consistent with an additive value function. We shall say that an ordinal preference attitude satisfies deterministic additive independence, or, simply, that it is additive, whenever some increasing transformation of its value function has the additive form

$$v(\underline{x}) = v_1(x_1) + \dots + v_n(x_n)$$

The following theorem and its corollary show that an additive ordinal preference attitude has important interpretations in terms of the

marginal value reduction coefficients $z_{ij}(\underline{x})$ and the marginal rates of substitution $\lambda_{ij}(\underline{x})$.

Theorem 3.9. An ordinal preference attitude is additive if and only if there are functions $z_1(x_1),\ldots,z_n(x_n)$ such that

$$z_{ij}(\underline{x}) = z_i(x_i)$$

for each i and $j \neq i$.

The functions $z_i(x_i)$ are used here to indicate that $z_{ij}(\underline{x})$ depends only on x_i. In Section 4.1 we extend this result to show how an additive value function can be constructed directly from the functions $z_i(x_i)$.

Corollary. An ordinal preference attitude is additive if and only if there are functions $g_1(x_1),\ldots,g_n(x_n)$ such that

$$\lambda_{ij}(\underline{x}) = \frac{g_i(x_i)}{g_j(x_j)}$$

for each i and j.

Proof of Theorem 3.9. The if part is proven in Appendix G. To prove the only if part, assume that

$$v(\underline{x}) = v_1(x_1) + \ldots + v_n(x_n) \tag{3.4}$$

Then, from Equation (2.1)

$$z_{ij}(\underline{x}) = -\frac{\dfrac{d^2v_i}{dx_i^2}}{\dfrac{dv_i}{dx_i}}$$

Note that this expression for $z_{ij}(\underline{x})$ holds for any value function that is an increasing transformation of (3.4). Therefore, let

$$z_i(x_i) = z_{ij}(\underline{x}) = -\frac{\dfrac{d^2 v_i}{dx_i^2}}{\dfrac{dv_i}{dx_i}}$$

Q.E.D.

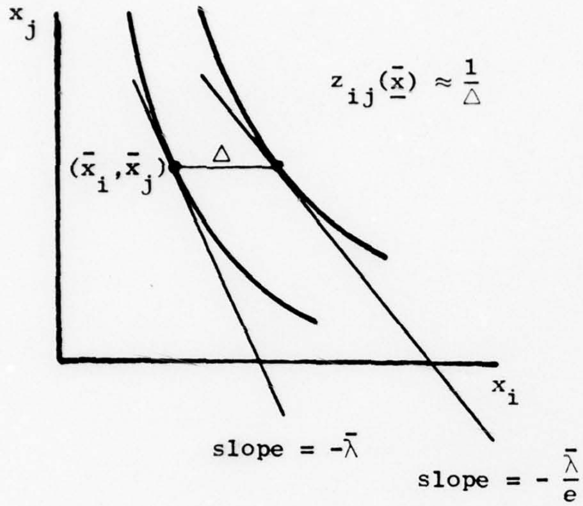
Theorem 3.9 and its corollary give us important interpretations for any additive ordinal preference attitude: first, from an increase of wealth in x_i, the percent change in the marginal value of x_i depends only on x_i; second, the decision maker's tradeoffs between x_i and x_j do not depend on his level of wealth in any other attribute. Figures 3.7a and 3.7b provide equivalent illustrations of this requirement in terms of the marginal value reduction coefficient and the marginal rates of substitution. The interpretation in Figure 3.7b is easily verified using the corollary and is discussed in detail in Keeney and Raiffa (1975).

In some cases it may not be obvious whether a value function can be written in an additive form. Theorem 3.9 would then be useful for establishing whether an additive form exists. For example, can the preference attitude expressed by

$$v(x_1,x_2) = 3x_1^2 + x_1^2 x_2 + 3x_1x_2 + 9x_1 + 2x_2 \qquad \text{for} \quad \begin{array}{l} x_1 > 0 \\ x_2 > 0 \end{array}$$

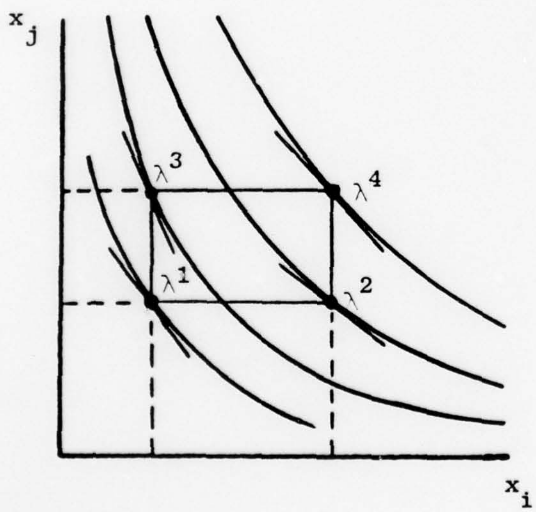
be written in an additive form? From Equation (2.1)

$$z_{ij}(\underline{x}) = -\frac{\dfrac{\partial^2 v}{\partial x_i^2}}{\dfrac{\partial v}{\partial x_i}} + \frac{\dfrac{\partial^2 v}{\partial x_i \partial x_j}}{\dfrac{\partial v}{\partial x_j}}$$



$v(\underline{x})$ is additive if and only if $z_{ij}(\underline{x})$ depends only on x_i for each i and j.

(a)



$v(\underline{x})$ is additive if and only if it is always true that

$$\frac{\lambda^1}{\lambda^2} = \frac{\lambda^3}{\lambda^4}$$

(b)

Figure 3.7. INTERPRETATIONS OF ADDITIVE VALUE FUNCTIONS (k IS A SUPERSCRIPT IN THE NOTATION λ^k).

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$$z_{12}(\underline{x}) = \frac{1}{x_1 + 1} + \frac{1}{x_1 + 2} - \frac{2}{2x_1 + 3} = z_1(x_1)$$

$$z_{21}(\underline{x}) = \frac{1}{x_2 + 3} = z_2(x_2)$$

Therefore, this ordinal preference attitude is additive. In fact, using Theorem 4.1 in the next chapter, one can actually find the additive form:

$$\hat{v}(x_1,x_2) = \ln (x_1 + 1) + \ln (x_1 + 2) + \ln (x_2 + 3)$$

It is easy to show that $\hat{v}(x_1,x_2)$ is an increasing transformation of $v(x_1,x_2)$.

Surprisingly, when $n \geq 3$ additive value functions are easier to interpret. Essentially, when the indifference curves between each pair of attributes are not influenced by the decision maker's wealth in any of the other attributes, his value function can be shown to be additive.

<u>Definition 3.3</u>. Deterministic independence. Let $n \geq 3$. A pair of attributes (x_i,x_j) is deterministically independent of its complement $\underline{x}_{ij}$ if for any $\bar{x}_i$, $\bar{x}_j$, $\hat{x}_i$, and $\hat{x}_j$

$$(\bar{x}_i,\bar{x}_j) \succ (\hat{x}_i,\hat{x}_j) \qquad \text{for some fixed } \underline{x}_{ij}$$

implies that

$$(\bar{x}_i,\bar{x}_j) \succ (\hat{x}_i,\hat{x}_j) \qquad \text{for any } \underline{x}_{ij}$$

In words, the preference ordering over (x_i,x_j) is independent of the fixed levels of the other attributes.

Note that utility independence implies deterministic independence: if (x_i, x_j) is utility independent of its complement, then the preference ordering over lotteries in (x_i, x_j) is independent of the fixed level of $\underline{x}_{ij}$; since this includes, as a special case, lotteries that specify a value for (x_i, x_j) with certainty, utility independence implies deterministic independence.

The following fundamental theorem is due to Debreu (1960) and, as stated below, depends also on results by Gorman (1968). A formal proof can be found in Krantz et al (1971). We must assume here that at least three attributes of $\underline{x}$ are essential in the sense that they can influence the ranking of outcomes.

Theorem 3.10 (Debreu). For $n \geq 3$, an ordinal preference attitude is additive if and only if for all $i \neq j$ each pair of attributes (x_i, x_j) is deterministically independent of its complement.

The striking aspect of this theorem is that for $n \geq 3$ we can establish additivity by simply requiring that the indifference curves between each pair of attributes do not depend on the levels of the other attributes. More complicated interpretations, such as those in Figure 3.7, are certainly implied by additivity but are not required to establish it.

Definition 3.3 and Theorem 3.10 are also valid in the general case where each attribute is itself a vector of attributes. For example, if each pair of attributes in (y_1, y_2, y_3) is deterministically independent of its complement where $y_1 = x_1$, $y_2 = (x_2, x_3)$, and $y_3 = (x_4, x_5, x_6)$, then we know from Theorem 3.10 that

$$v(\underline{x}) = v_1(x_1) + v_2(x_2,x_3) + v_3(x_4,x_5,x_6)$$

3.6 Independence Hierarchy

The functional forms implied by the three independence conditions that we have discussed are related through a simple hierarchy: risk additive independence implies mutual utility independence; mutual utility independence implies deterministic additive independence. This is illustrated in Figure 3.8.

A more detailed independence hierarchy is discussed by Fishburn and Keeney (1974).

Risk Additive Independence

$$u(\underline{x}) = u_1(x_1) + \ldots + u_n(x_n)$$

$$\downarrow$$

Mutual Utility Independence

$$u(\underline{x}) = \begin{cases} u_1(x_1) + \ldots + u_n(x_n) \\ \text{or} \\ u_1(x_1)\, u_2(x_2) \ldots u_n(x_n) \end{cases}$$

$$\downarrow$$

Deterministic Additive Independence

$$v(\underline{x}) = v_1(x_1) + \ldots + v_n(x_n)$$

Figure 3.8. INDEPENDENCE HIERARCHY.

Chapter 4

ASSESSMENT PROCEDURE

In this chapter we show how the theory in Chapters 2 and 3 can be applied to assess a decision maker's preferences. The integral notation used in Sections 4.1 and 4.2 is defined in Appendix A.

4.1 Assessing Additive Value Functions

By modeling the decision maker's marginal value reduction coefficients, we can generate a functional form for his ordinal preference attitude. The idea underlying this approach is introduced by Barrager (1975). The following result is an important extension of Theorem 3.9.

Theorem 4.1. A decision maker has marginal value reduction coefficients $z_{ij}(\underline{x}) = z_i(x_i)$ for each i and $j \neq i$ if and only if his value function can be written

$$v(\underline{x}) = a_1 \int e^{-\int z_1(x_1)} + \dots + a_n \int e^{-\int z_n(x_n)}$$

Corollary. A value function is additive if and only if for all i and j

$$\lambda_{ij}(\underline{x}) = \frac{a_i}{a_j} e^{-\int z_i(x_i) + \int z_j(x_j)}$$

Theorem 4.1 is important because if we can model the decision maker's marginal value reduction coefficients with functions $z_1(x_1), \dots, z_n(x_n)$, then it provides us with a corresponding additive value function. Further, if the marginal value reduction coefficients cannot be modeled in this way, then Theorem 3.9 guarantees that no additive value function is appropriate.

Proof of Theorem 4.1. For the *if* part

$$\frac{\partial v}{\partial x_i} = a_i \, e^{-\int z_i(x_i)}$$

$$\frac{\partial^2 v}{\partial x_i^2} = -a_i z_i(x_i) \, e^{-\int z_i(x_i)}$$

$$\frac{\partial^2 v}{\partial x_i \partial x_j} = 0 \qquad \text{for} \quad i \neq j$$

Using Equation (2.1)

$$z_{ij}(\underline{x}) = - \frac{\dfrac{\partial^2 v}{\partial x_i^2}}{\dfrac{\partial v}{\partial x_i}} + \frac{\dfrac{\partial^2 v}{\partial x_i \partial x_j}}{\dfrac{\partial v}{\partial x_j}} = z_i(x_i) \qquad \text{for} \quad i \neq j$$

To prove the *only if* part, assume that we have functions $z_i(x_i)$ such that

$$z_i(x_i) = z_{ij}(\underline{x}) \qquad \text{for all} \ \underline{x}, \ i, \ \text{and} \ j \neq i$$

Then from Theorem 3.9, $v(\underline{x})$ can be written

$$v(\underline{x}) = v_1(x_1) + \dots + v_n(x_n)$$

Thus, Equation (2.1) becomes

$$- \frac{\dfrac{d^2 v_i}{dx_i^2}}{\dfrac{dv_i}{dx_i}} = z_i(x_i)$$

$$\frac{d}{dx_i} \ln \frac{dv_i}{dx_i} = -z_i(x_i)$$

$$\ln \frac{dv_i}{dx_i} = -\int z_i(x_i)$$

$$\frac{dv_i}{dx_i} = a_i \, e^{-\int z_i(x_i)}$$

Thus, $v_i(x_i)$ can be written

$$v_i(x_i) = a_i \int e^{-\int z_i(x_i)}$$

where a_i is a constant of integration. Q.E.D.

Therefore, in general, we can use the following procedure to assess an additive value function. Assess the decision maker's marginal value reduction coefficients at several points. Use this data to model each marginal value reduction coefficient with an appropriate function $z_i(x_i)$. Using Theorem 4.1 $v(\underline{x})$ is thus determined up to the constants a_i. To assess these constants, let $a_1 = 1$ and choose a nominal wealth level $\bar{\underline{x}}$. Then, for each i, assess the marginal rate of substitution

$$\lambda_{i1}(\underline{x}) = a_i \, e^{-\int z_i(\bar{x}_i)+\int z_j(\bar{x}_j)}$$

and solve for a_i.

We illustrate this procedure by generating three simple value functions.

<u>Linear Value Function</u>

If

$$z_{ij}(\underline{x}) = 0 \qquad \text{for all } i \text{ and } j$$

then from Theorem 4.1

$$v(\underline{x}) = a_1 x_1 + \ldots + a_n x_n$$

To find the constants a_i, let $a_1 = 1$ and assess $\lambda_{i1}(\underline{x})$ at a nominal wealth level $\bar{\underline{x}}$.

$$\lambda_{i1}(\bar{\underline{x}}) = a_i$$

The linear value function is often used for local modeling of cash flows (see Section 2.6) and applies whenever the decision maker is willing to trade off attributes at a fixed price within the region of interest.

Sum-of-Exponentials Value Function

When

$$z_{ij}(\underline{x}) = \gamma_i \qquad \text{for all } i \neq j$$

the percent decrease in the marginal value $\lambda_{ij}(\underline{x})$ per unit increase of x_i is independent of the decision maker's wealth. Applying Theorem 4.1

$$v(\underline{x}) = v_1(x_1) + \ldots + v_n(x_n)$$

where

$$v_i(x_i) = \begin{cases} -a_i \, e^{-\gamma_i x_i} & \text{when } \gamma_i \neq 0 \\ a_i x_i & \text{when } \gamma_i = 0 \end{cases}$$

Sum-of-exponentials indifference curves are illustrated in Figure 4.1.

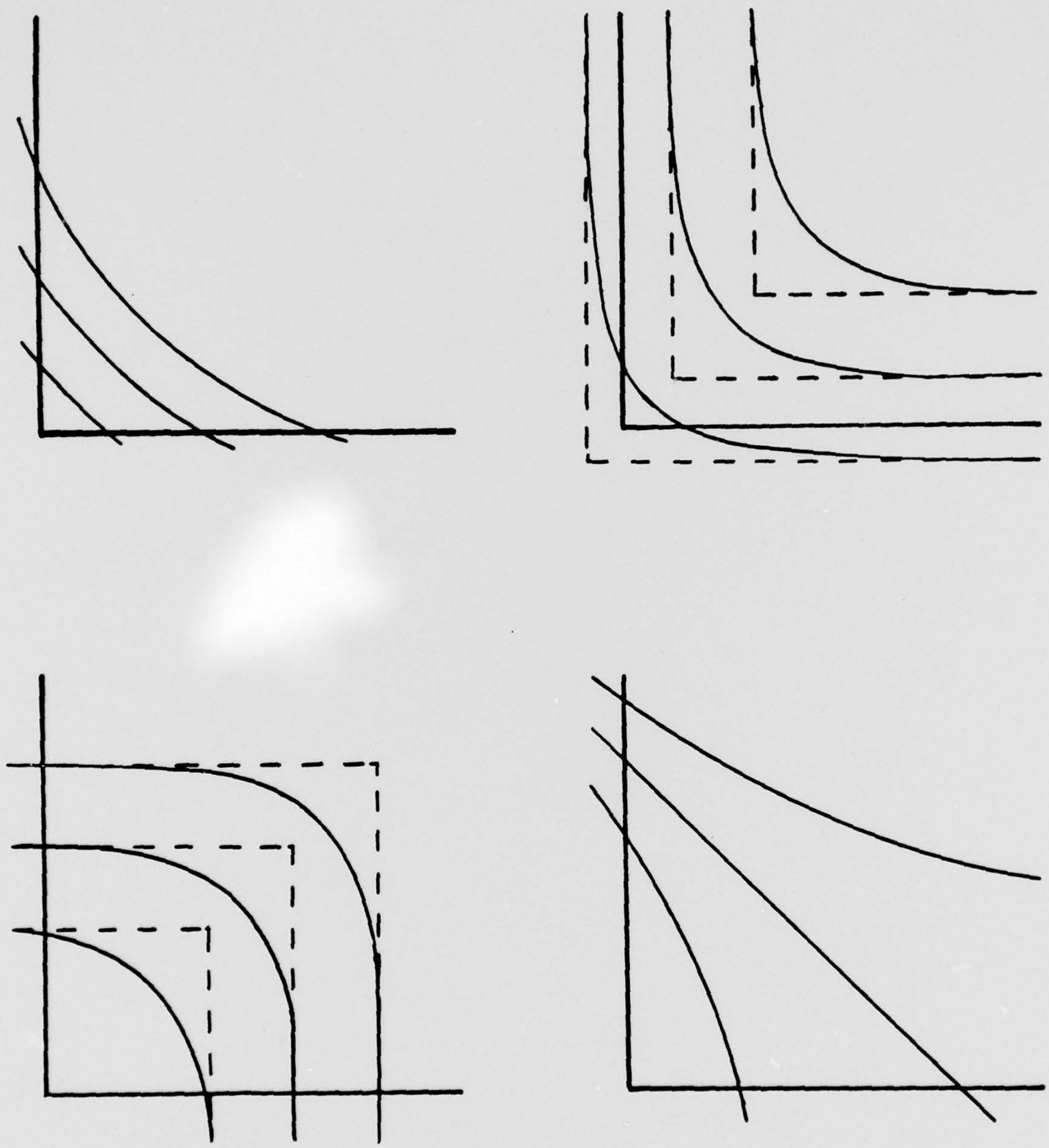

Figure 4.1. SUM-OF-EXPONENTIALS INDIFFERENCE CURVES.

To assess this value function, first assess the marginal value reduction coefficients γ_i using Figure 2.7 or 2.8. Then let $a_1 = 1$, assess $\lambda_{i1}(\underline{x})$ at some nominal wealth level $\bar{\underline{x}}$, and solve for a_i.

$$\lambda_{i1}(\bar{\underline{x}}) = a_i \frac{\gamma_i}{\gamma_1} e^{-\gamma_i \bar{x}_i + \gamma_1 \bar{x}_1}$$

<u>Posynomial Value Function</u>

In many cases we would expect the decision maker to display increasing marginal value preservation tolerance. That is, as he becomes wealthier in an attribute, an additional unit of it leads to a smaller percent decrease in its marginal value. A simple way to capture this notion is to specify

$$z_{ij}(\underline{x}) = \frac{1 + \alpha_i}{x_i} \qquad \text{for } i \neq j$$

Using Theorem 4.1

$$v(\underline{x}) = v_1(x_1) + \ldots + v_n(x_n)$$

where

$$v_i(x_i) = \begin{cases} -a_i x_i^{-\alpha_i} & \text{when } \alpha_i \neq 0 \\ a_i \ln x_i & \text{when } \alpha_i = 0 \end{cases}$$

To assess this value function, start at some nominal wealth level $\bar{\underline{x}}$, use Figure 2.7 to assess the marginal value reduction coefficients

$$z_{ij}(\bar{\underline{x}}) = \frac{1 + \alpha_i}{\bar{x}_i} \qquad \text{for } i = 1, \ldots, n$$

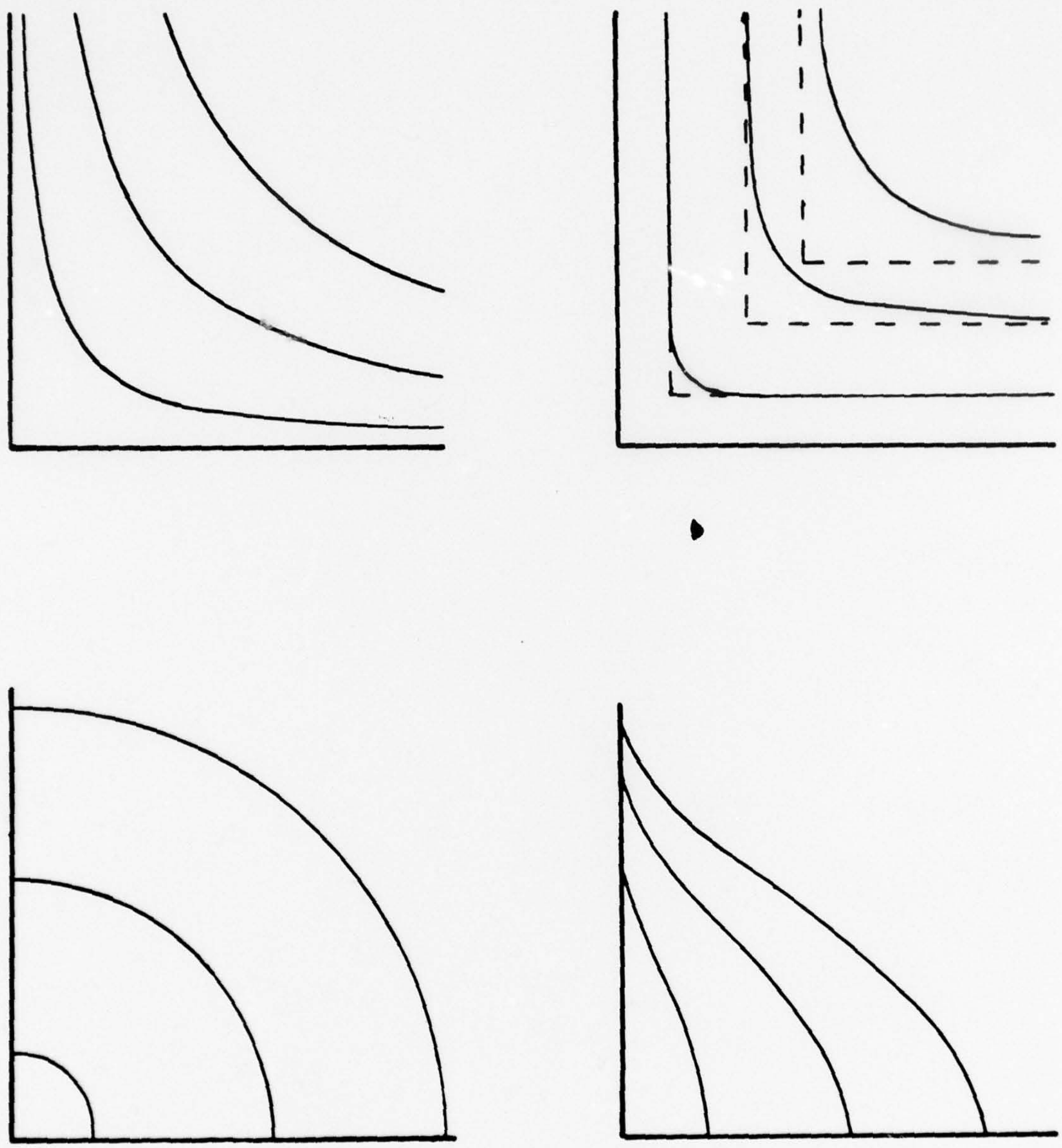

Figure 4.2. POSYNOMIAL INDIFFERENCE CURVES.

and solve for each α_i. Then, let $a_1 = 1$, assess

$$\lambda_{i1}(\underline{\bar{x}}) = a_i \frac{\alpha_i \bar{x}_i^{-\alpha_i - 1}}{\alpha_1 \bar{x}_1^{-\alpha_1 - 1}} \qquad \text{for } i = 2, \ldots, n$$

and solve for each a_i.

4.2 Assessing Risk Preference in One Dimension

Pratt's risk aversion coefficient, discussed in Section 2.3, embodies all of the essential information in a utility function. The following theorem shows how a one-dimensional utility function can be generated from the risk aversion coefficient.

Theorem 4.2 (Pratt). If a decision maker has risk aversion coefficient $r(x)$, then his utility function can be written

$$u(x) = \int e^{-\int r(x)}$$

One can prove this theorem by solving the differential equation

$$r(x) = - \frac{\frac{d^2u}{dx^2}}{\frac{du}{dx}}$$

If we can model the decision maker's risk aversion coefficient $r(x)$, then Theorem 4.2 provides us with the corresponding utility function. Some particular forms for $r(x)$ lead to delta property interpretations and imply that $u(x)$ has a linear, exponential, logarithmic, or power form. These are provided by Theorem 3.4 $(n = 1)$.

Theorems 4.1 and 4.2 give us a general technology for implementing the two-step procedure (Figure 2.4) when ordinal preferences are additive.

4.3 Multiattribute Preference Assessment Flow Chart

In this section we assemble the technology that we have developed into a procedure for assessing multiattribute preferences. See Figures 4.3a, 4.3b, and 4.3c.

We begin by asking the decision maker whether he is always indifferent between L_1 and L_2. If so, he is consistent with risk additive independence, and we can check, for each attribute, to see whether one of the delta properties for additive utility functions holds. If so, Theorem 3.4 applies and $u_i(x_i)$ is the appropriate linear, exponential, logarithmic, or power form as shown. If $u_i(x_i)$ does not satisfy a delta property, we can generate a form for it by modeling $z_i(x_i)$ or $r_i(x_i)$ (they are equal by Theorem 3.3) and applying Theorem 4.1 or 4.2. After this is done for each i, the scaling constants a_i can be found by assessing the marginal rates of substitution $\lambda_{i1}(\bar{\underline{x}})$.

To check for consistency we could start with mutual utility independence and deterministic additive independence, which are both implied by risk additive independence. Then, we could proceed to check consistency locally using the preference measures that we did not assess directly; in this case, they are the marginal rate of substitution $\lambda_{ij}(\underline{x})$, the substitution aversion coefficient $s_{ij}(\underline{x})$, and the numeraire risk aversion coefficient $r_\nu(\underline{x})$, the last of which we can derive after choosing a numeraire.

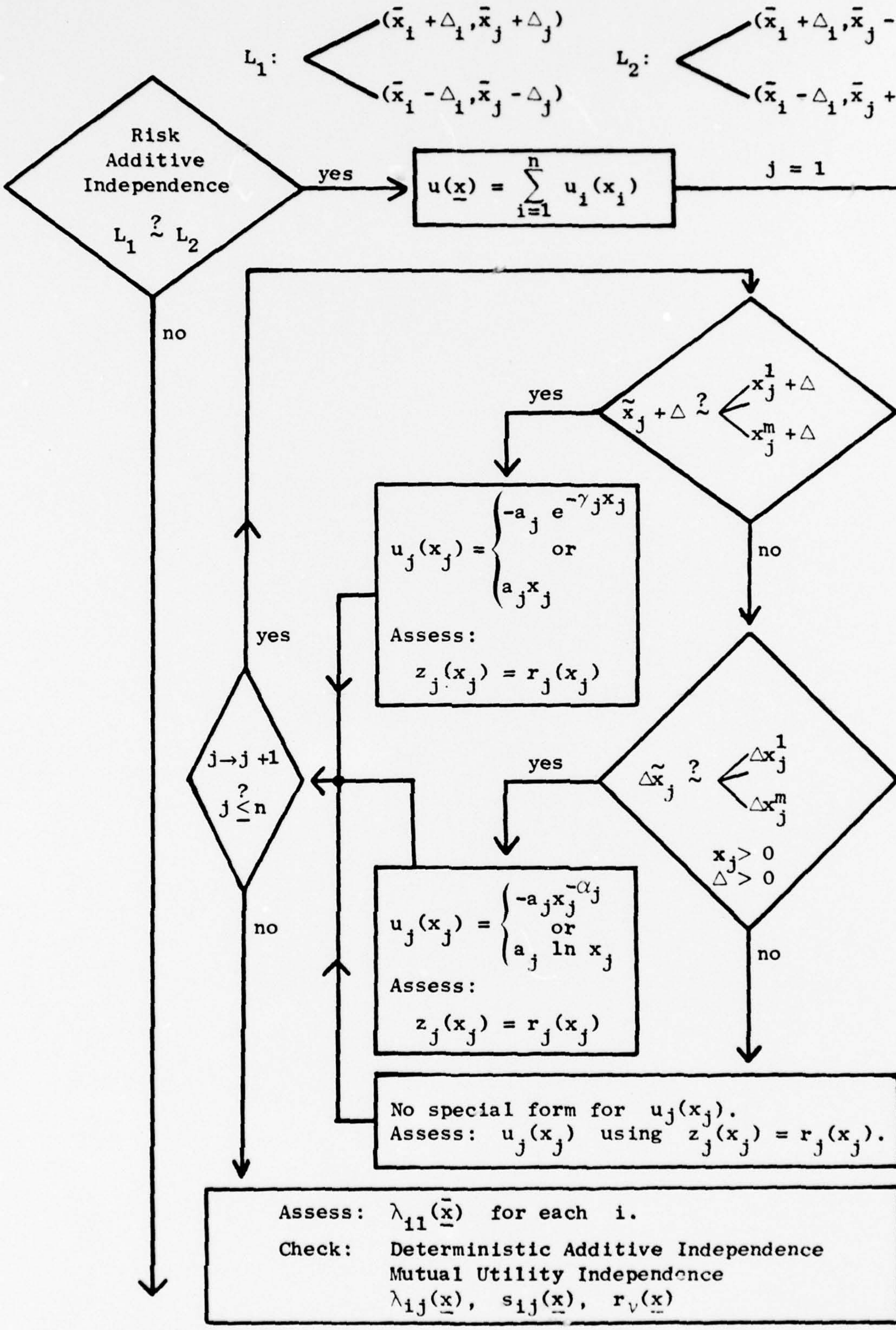

Figure 4.3a. *MULTIATTRIBUTE PREFERENCE ASSESSMENT FLOW CHART.*

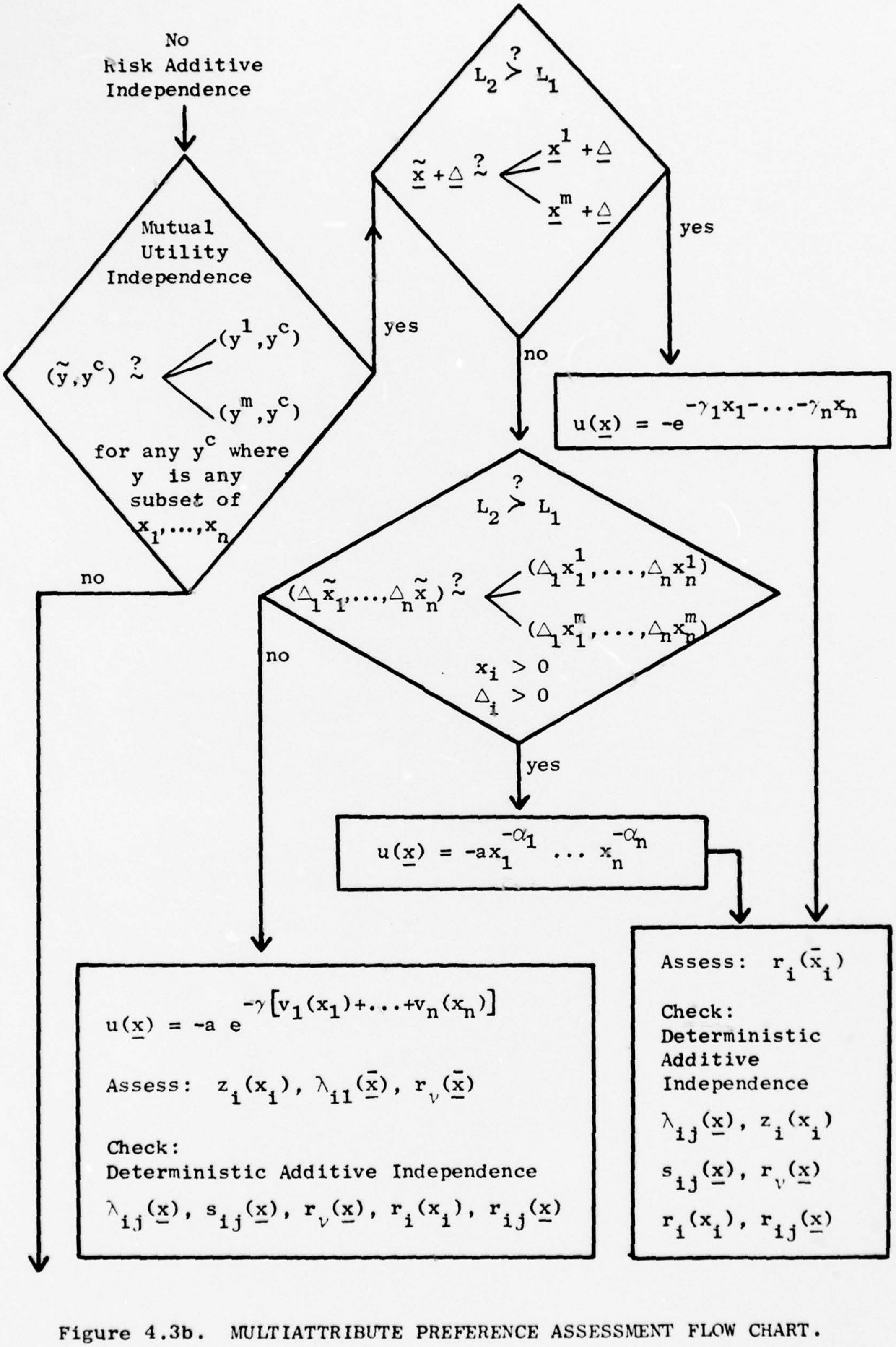

Figure 4.3b. MULTIATTRIBUTE PREFERENCE ASSESSMENT FLOW CHART.

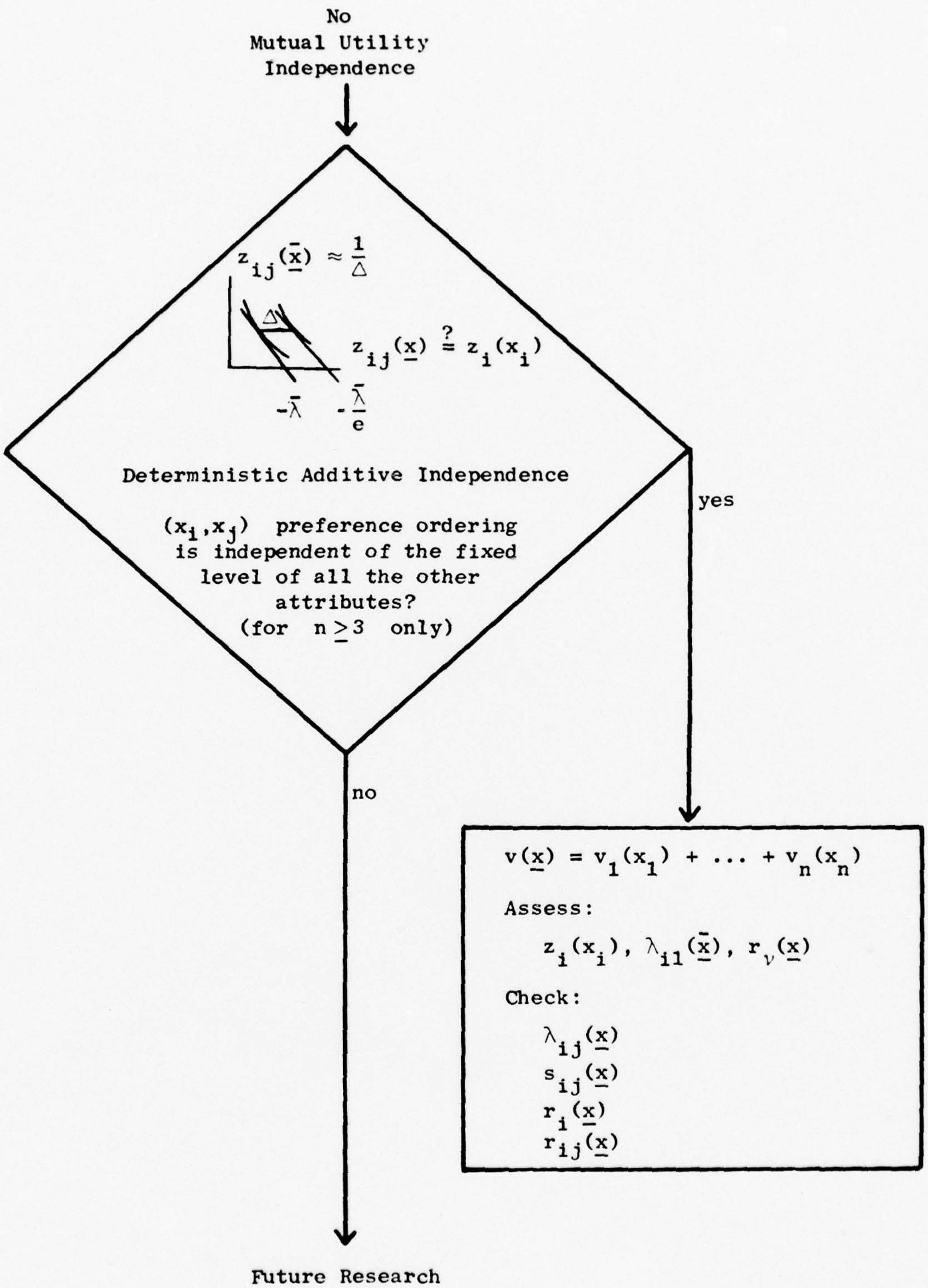

Figure 4.3c. MULTIATTRIBUTE PREFERENCE ASSESSMENT FLOW CHART.

If the decision maker does not satisfy risk additive independence, we proceed to Figure 4.3b and determine whether he satisfies mutual utility independence. If so, his utility function is multiplicative since risk additive independence is not satisfied. When $n = 2$ we can check for mutual utility independence directly using Figure 3.4. When $n \geq 3$ it would generally be easier to use the corollary to Theorem 3.6.

If, in addition to mutual utility independence, the decision maker satisfies one of the multiplicative delta properties and is risk averse, then the theorems of Section 3.4 apply, and his utility function is either a product-of-exponentials or a product-of-powers. The parameters for these forms can be determined using the single-attribute risk aversion coefficient at a nominal wealth level. Then deterministic additive independence and all six preference measures can be used to check for consistency.

If neither multiplicative delta property is appropriate, we still know from Theorem 3.6 that the multiplicative utility function can be written as an exponential function of the decision maker's additive value function. To assess this form, first assess the additive value function using $z_i(x_i)$ and $\lambda_{i1}(\bar{\underline{x}})$ as described in Section 4.1. Then, considering $v(\underline{x})$ itself as a numeraire, assess the numeraire risk aversion coefficient $r_v(\underline{x}) = \gamma$ at a nominal wealth level $\bar{\underline{x}}$ and check for consistency as shown.

If mutual utility independence is not satisfied, we proceed to Figure 4.3c and try deterministic additive independence. If either of the two conditions shown is satisfied, the decision maker's ordinal preference function is additive, and we assess his preferences using the two-step procedure: first, assess the additive value function using $z_i(x_i)$

and $\lambda_{i1}(\bar{\underline{x}})$ as described in Section 4.1; second, choose an appropriate numeraire and assess risk preference over it as discussed in Section 4.2.

If deterministic additive independence is not satisfied, we know that no additive value function is appropriate. This is an area for future research discussed in Chapter 5.

Extensions

The multiattribute preference assessment flow chart in its present form is just a beginning. We look forward to seeing it expanded and improved through practical experience and additional research.

One possible extension is to treat explicitly the cases in which an independence condition is satisfied for some subsets of attributes but not for others. As noted in Section 3.1, if y_1 and y_2 are risk additive independent where $y_1 = (x_1,x_2)$ and $y_2 = (x_3,x_4)$, then

$$u(\underline{x}) = u_1(x_1,x_2) + u_2(x_3,x_4)$$

We could then apply the flow chart, starting with Figure 4.3a, to (x_1,x_2) and (x_3,x_4) separately. We might find, for example: that (x_1,x_2) are mutually utility independent but not risk additive independent; and that (x_3,x_4) satisfy deterministic additive independence but not mutual utility independence. This would imply

$$u(\underline{x}) = -a\ e^{-\gamma(v_1(x_1)+v_2(x_2))} + u_3(v_3(x_3) + v_4(x_4))$$

The two terms of this equation can be assessed using the bottom boxes of Figures 4.3b and 4.3c, respectively. It is easy to show, however, that the overall ordinal preference attitude $v(x_1,x_2,x_3,x_4)$ cannot be written in an additive form.

Whether we should expand the flow chart to handle explicitly this kind of situation depends on whether it would arise in practical application. For now, we shall leave it in its present simplified form.

Another possible extension arises when each attribute is utility independent of its complement but the attributes are not mutually utility independent. When $n \geq 3$, this condition implies a multilinear utility function (see Keeney and Raiffa (1975)). The value function implied by this condition has a restricted, but generally nonadditive, form. Again, we shall wait for experience to indicate whether assessment methods for this type of preference attitude warrant further investigation.

4.4 Planning a Sierra Backpacking Vacation

To illustrate the preference assessment flow chart we now assess a multiattribute utility function for an actual decision maker.

Steve Haas, a 29 year old systems analyst living near Palo Alto, is an experienced and enthusiastic backpacker. Steve has recently accepted responsibility for planning a one-week backpacking trip for himself and several other people. Of the many decisions involved in this planning, his most important one is choosing exactly where to go. We shall assess Steve's preferences over the possible outcomes of this decision.

Describing the Outcomes

In the Sierra Nevada within five to seven hours driving time of Palo Alto, there are numerous national forests and wilderness areas, and three national parks: Sequoia, Kings Canyon, and Yosemite. During

an initial review of alternatives, Steve mentioned the attractive and unattractive aspects of each. His foremost considerations were the number of other backpackers in the area, the number of lakes in the area, the possible weather conditions, and the area's accessibility by car. Other considerations such as majestic beauty and the number of trails in the area were also mentioned; however, these qualities did not differ appreciably for the alternatives under consideration and, thus, were not important to the planning decision.

The feeling of peacefulness and solitude in an area of majestic beauty is one of the joys of Sierra backpacking. When the trails and campsites are crowded with other backpackers, this pleasant sensation is greatly diminished. A simple measure of backpacker density, which is easy for the decision maker to visualize, is the number of backpackers per square mile. So that more is preferred to less, it is convenient to define the reciprocal of this quantity, the number of square miles per backpacker, as the first attribute. This is defined precisely in Figure 4.4. Limited advance information about the value of x_1 in any area is available from park rangers.

Areas that have many lakes are preferable to those that have few. Sierra lakes not only have aesthetic appeal, but also provide drinking water, campsites, and, often, excellent fishing. The number of lakes per square mile is the second attribute in Figure 4.4. The value of this attribute for any area can be determined in advance by examining topographical maps published by the U. S. Geological Survey.

The weather conditions can be the most important determinant of comfort and safety in wilderness backpacking. Precipitation, either rain or snow, is particularly unpleasant, but the decision maker's

$x_1 = \frac{a}{n}$ number of square miles per backpacker

$x_2 = \frac{l}{a}$ number of lakes per square mile

$x_3 = t$ lowest nighttime temperature

$x_4 = -c$ transportation cost per person

where

a = area in square miles within one mile of the proposed trail route

l = number of lakes with at least 10,000 square feet of surface area within hiking area a

n = number of backpackers in hiking area a but not in the decision maker's group at 6 p.m. on the third day of hiking (a representative day and time)

t = lowest temperature in degrees Fahrenheit at the decision maker's campsite during the trip

c = ((auto expenses in dollars per mile) $\times$ (round trip miles) $\div$ (number of passengers))+(value of a passenger's time in dollars per passenger hour) $\times$ (round trip driving hours - 8 hours)

Figure 4.4. FOUR ATTRIBUTES FOR PLANNING A BACKPACKING TRIP.

knowledge about the possibility of precipitation did not vary from alternative to alternative.

The temperature, however, can be expected to decrease by three or four degrees per 1,000 feet increase in elevation. Low nighttime temperatures are uncomfortable and, in extreme cases, dangerous. We model this explicitly with a third attribute, the lowest temperature at the decision maker's campsite for the duration of the trip. When the planning decision is made, x_3 is, of course, uncertain.

Transportation cost per person, the fourth attribute, includes both automobile expenses and the value of a passenger's time as defined in Figure 4.4. A passenger's time is valued only in excess of 8 hours since any round trip to the Sierras would take at least that long. We define x_4 as the negative cost per person so that larger values of x_4 are preferred to smaller ones. For any alternative, x_4 is uncertain since round trip driving time can only be estimated roughly in advance.

Steve's current automobile expenses are $0.09 per mile, and for the upcoming trip he values passenger time at $4.33 per passenger hour.

Steve agreed that $\underline{x} = (x_1, x_2, x_3, x_4)$ adequately describes the possible outcomes: that is, if he knew in advance only the value of $\underline{x}$ for each alternative, he would be satisfied to make a decision on that basis. He mentioned in addition that his preferences among these four attributes might not change significantly over time and that an accurate preference function could also be useful to him in future backpacking decisions.

To familiarize the decision maker with numerical values of these attributes, we discussed his experiences on several previous Sierra backpacking trips; some specific outcomes appear in Table 4.1. Then, the decision maker assigned probabilities to the outcomes of some present

Table 4.1

THREE PREVIOUS BACKPACKING OUTCOMES

Trip	x_1 Sq. Mi./ Backpacker	x_2 Lakes/ Sq. Mi.	x_3 Temperature (Fahrenheit)	x_4 Cost (Dollars)
Chain Lakes	1	0.31	29	-37.5
Desolation Valley	1/5.5	1.3	35	-22.1
Emigrant Wilderness	1/6	1.4	34	-12.3

Table 4.2

FOUR PRESENT ALTERNATIVES

Trip	x_1			x_2			x_3			x_4		
	$f_{0.1}$	$f_{0.5}$	$f_{0.9}$	$f_{0.1}$	$f_{0.5}$	$f_{0.9}$	$f_{0.1}$	$f_{0.5}$	$f_{0.9}$	$f_{0.1}$	$f_{0.5}$	$f_{0.9}$
Granite Dome	1/5	1/2.7	1/1.5	1.4	1.4	1.4	26	30	36	-19.0	-15.5	-13.0
Lake of the Fallen Moon	1/9	1/6	1/3	0.82	0.82	0.82	25	29	34	-30.0	-25.6	-22.0
Middle Fork Bishop Creek	1/7.5	1/4	1/2	1.5	1.5	1.5	22	28	33	-50.0	-43.5	-39.0
Woodchuck Creek	1/4	1/2.5	1	0.35	0.35	0.35	28	32	37	-42.0	-37.5	-34.0

alternatives: for each attribute in Table 4.2, f_1 is the 1 fractile of that attribute's marginal distribution.

Limiting Functional Form

Proceeding according to the flow chart in Figure 4.3a, we asked the decision maker to supply certain equivalents for lotteries L_1 and L_2 in Figures 4.5a and 4.5b. Answers supplied by the decision maker are underlined. In both cases L_2 is preferred to L_1; subsequent questions verified that L_2 would be preferred for any pair of attributes over a broad range. Therefore, the decision maker is multiattribute risk averse, and no additive utility function is appropriate.

The next step in the flow chart, Figure 4.3b, is to check for mutual utility independence. For four attributes we use the corollary to Theorem 3.6: first check for deterministic independence; second, if the attributes are deterministically independent, test whether x_1 is utility independent of its complement.

To test for deterministic independence we asked the questions in Tables 4.3a and 4.3b. In the top box of Table 4.3a, for example, questions proceeded like this:

"Suppose that cost is $25 per person and that lowest nighttime temperature is 32 degrees. Then 4 backpackers/sq. mi. and 0.7 lakes/sq. mi. is indifferent to 5 backpackers and how many lakes?"

"About 0.85."

"O.K. Now suppose that temperature rises to 35. Would your answer change?"

"I don't think so."

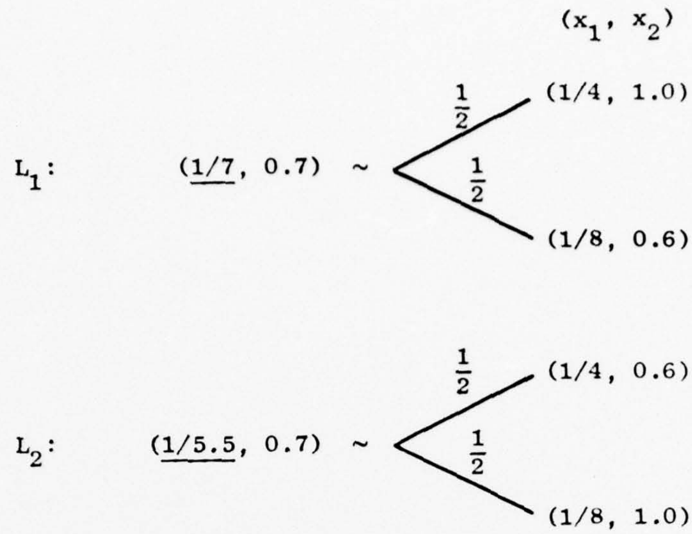

Figure 4.5a. TESTING FOR AN ADDITIVE UTILITY FUNCTION $(x_3, x_4) = (32, -25)$.

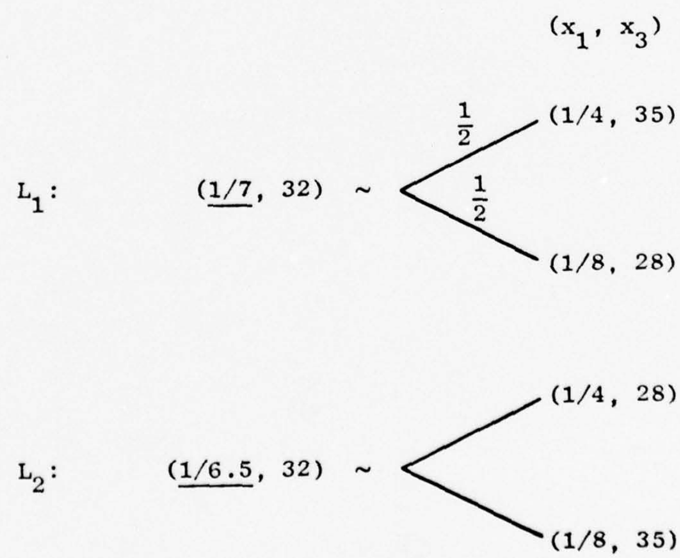

Figure 4.5b. TESTING FOR AN ADDITIVE UTILITY FUNCTION $(x_2, x_4) = (0.7, -25)$.

Table 4.3a

TESTING FOR DETERMINISTIC INDEPENDENCE

(x_1, x_2) $(1/4, 0.7) \sim (1/5, \underline{0.85})$ when $(x_3, x_4) = (32, -25)$. Would your answer change if $(x_3, x_4) = (35, -50)$? <u>No</u>. Would your answer change if $(x_3, x_4) = (25, -15)$? <u>No</u>.	Do your tradeoffs between backpackers and lakes depend on temperature and cost? <u>No</u>.
(x_1, x_3) $(1/4, 32) \sim (1/5, \underline{37})$ when $(x_2, x_4) = (0.7, -25)$. Would your answer change if $(x_2, x_4) = (1.0, -40)$? <u>No</u>. Would your answer change if $(x_2, x_4) = (0.5, -15)$? <u>No</u>.	Do your tradeoffs between backpackers and temperature depend on lakes and cost? <u>No</u>.*
(x_2, x_3) $(0.7, 32) \sim (0.8, \underline{29.5})$ when $(x_1, x_4) = (1/4, -25)$. Would your answer change if $(x_1, x_4) = (1/7, -40)$? <u>No</u>. Would your answer change if $(x_1, x_4) = (1/2, -15)$? <u>No</u>.	Do your tradeoffs between lakes and temperature depend on backpackers and cost? <u>No</u>.*

*Valid when there are no more than 12 backpackers/lake.

Table 4.3b

TESTING FOR DETERMINISTIC INDEPENDENCE

(x_1, x_4) $(1/4, -30) \sim (1/5, \underline{-21})$ when $(x_2, x_3) = (0.7, 32)$. Would your answer change if $(x_2, x_3) = (0.6, 39)$? No. Would your answer change if $(x_2, x_3) = (1.0, 23)$? No.	Do your tradeoffs between backpackers and cost depend on lakes and temperature? No.*
(x_2, x_4) $(0.7, -15) \sim (0.9, \underline{-25})$ when $(x_1, x_3) = (1/5, 32)$. Would your answer change if $(x_1, x_3) = (1/8, 25)$? No. Would your answer change if $(x_1, x_3) = (1/2, 40)$? No.	Do your tradeoffs between lakes and cost depend on backpackers and temperature? No.*
(x_3, x_4) $(36, -30) \sim (32, \underline{-22})$ when $(x_1, x_2) = (1/5, 0.7)$. Would your answer change if $(x_1, x_2) = (1/2, 0.3)$? No. Would your answer change if $(x_1, x_2) = (1/9, 1.3)$? No.	Do your tradeoffs between temperature and cost depend on backpackers and lakes? No.

*Valid when there are no more than 12 backpackers/lake.

"O.K. And if cost rises to \$50?"

"That wouldn't make any difference."

"O.K. Would it make any difference if temperature drops to 25 and cost to \$15?"

"No, it wouldn't. I don't think that my tradeoffs between backpackers and lakes have anything to do with temperature and cost."

"O.K. Fine."

Thus, we verified that (x_1,x_2) is deterministically independent of (x_3,x_4). With similar questions we found each of the other five pairs of attributes deterministically independent of its complement over a broad region.

The decision maker noted, however, that several of the independence conditions would not be satisfied if there were many backpackers and few lakes: lake campsites are limited, and overcrowded campsites are particularly unpleasant. In particular, the decision maker reiterated that $(1/4, 0.7, 32, -25) \sim (1/4, 0.8, \underline{29.5}, -25)$ but noted that if we increase the number of backpackers/sq. mi. to 14, there would be roughly $14/0.7 = 20$ backpackers/lake. In this case he could envision overcrowded campsites and, thus, would be willing to give up more temperature for additional lakes: $(1/14, 0.7, 32, -25) \sim (1/14, 0.8, \underline{27}, -25)$.

Examining this effect carefully, we found that deterministic independence is satisfied when there are no more than 12 backpackers/lake, i.e., when $x_1x_2 \geq 1/12$. For the decision at hand this presents no difficulty since 12 backpackers/lake is a very bad outcome, which would be avoided by consulting the park rangers in advance. This effect, however, might be a good starting point for a researcher interested in developing techniques for assessing nonadditive value functions.

Assuming $x_1 x_2 \geq 1/12$ pairwise deterministic independence is satisfied, and from Theorem 3.10

$$v(\underline{x}) = v_1(x_1) + v_2(x_2) + v_3(x_3) + v_4(x_4)$$

With the questions in Figure 4.6 we proceeded to verify that x_1 is utility independent of (x_2, x_3, x_4). Therefore, from the corollary to Theorem 3.6, attributes x_1, x_2, x_3, and x_4 are mutually utility independent.

A quick check of the delta properties for multiplicative utility functions revealed that neither was appropriate. Thus, we arrived at the bottom box in the multiattribute preference assessment flow chart, Figure 4.3b, and the decision maker's utility function is limited to the form

$$u(\underline{x}) = -a\, e^{-\gamma(v_1(x_1)+v_2(x_2)+v_3(x_3)+v_4(x_4))} \tag{4.1}$$

Assessing Ordinal Preferences

To assess (4.1) we first assess the value function $v(\underline{x})$. Our strategy is to model the marginal value reduction coefficients $z_{ij}(\underline{x}) = z_i(x_i)$, and then apply the technology in Section 4.1.

For estimating these coefficients we use the 17 pairs of indifference points, labeled A through Q, in Table 4.4. Corresponding to these assessments, the decision maker is indifferent between the endpoints of each solid line segment in Figures 4.7a and 4.7b. We interpret the negative slope of each line segment in Table 4.4 as an estimate of the local marginal rate of substitution. Using this information we shall estimate some particular values of $z_1(x_1)$, $z_2(x_2)$, $z_3(x_3)$, and $z_4(x_4)$.

x_1 certain equivalent	lottery on x_1	level of (x_2, x_3, x_4)
$\underline{1/4.5}$ $\sim$	$\frac{1}{2}$ 1/2 ; $\frac{1}{2}$ 1/6	(0.7, 32, -25)
$\underline{1/4.5}$ $\sim$	$\frac{1}{2}$ 1/2 ; $\frac{1}{2}$ 1/6	(0.5, 25, -15)
$\underline{1/4.5}$ $\sim$	$\frac{1}{2}$ 1/2 ; $\frac{1}{2}$ 1/6	(1.2, 35, -40)

Does your certain equivalent for any lottery on x_1 depend on the level of (x_2, x_3, x_4)? <u>No</u>.

Figure 4.6. TESTING WHETHER x_1 IS UTILITY INDEPENDENT OF (x_2, x_3, x_4).

Table 4.4

ASSESSMENTS FOR ESTIMATING z_1, z_2, z_3, AND z_4

	Line Segment	Indifference Points	Negative Slope
(x_1,x_2) Indifference Points	A	(1/6, 1.0) ~ (1/7, <u>1.25</u>)	10.5
	B	(1/4, 1.0) ~ (1/5, <u>1.30</u>)	6.0
	C	(1/2, 1.0) ~ (1/3, <u>1.35</u>)	2.1
	D	(1/6, 0.70) ~ (1/7, <u>0.83</u>)	5.5
	E	(1/4, 0.70) ~ (1/5, <u>0.85</u>)	3.0
	F	(1/2, 0.70) ~ (1/3, <u>0.85</u>)	0.9
	G	(1/6, 0.50) ~ (1/7, <u>0.60</u>)	4.2
	H	(1/4, 0.50) ~ (1/5, <u>0.60</u>)	2.0
	I	(1/2, 0.50) ~ (1/3, <u>0.63</u>)	0.8
(x_1,x_3) Indifference Points	J	(1/7, 20) ~ (1/8, <u>22</u>)	0.0089
	K	(1/7, 25) ~ (1/8, <u>27.5</u>)	0.0071
	L	(1/7, 32) ~ (1/8, <u>37</u>)	0.0036
	M	(1/7, 34) ~ (1/8, <u>40</u>)	0.0030
(x_1,x_4) Indifference Points	N	(1/4, -50) ~ (1/5, <u>-41</u>)	0.0056
	O	(1/4, -40) ~ (1/5, <u>-31</u>)	0.0056
	P	(1/4, -30) ~ (1/5, <u>-21</u>)	0.0056
	Q	(1/4, -20) ~ (1/5, <u>-11</u>)	0.0056

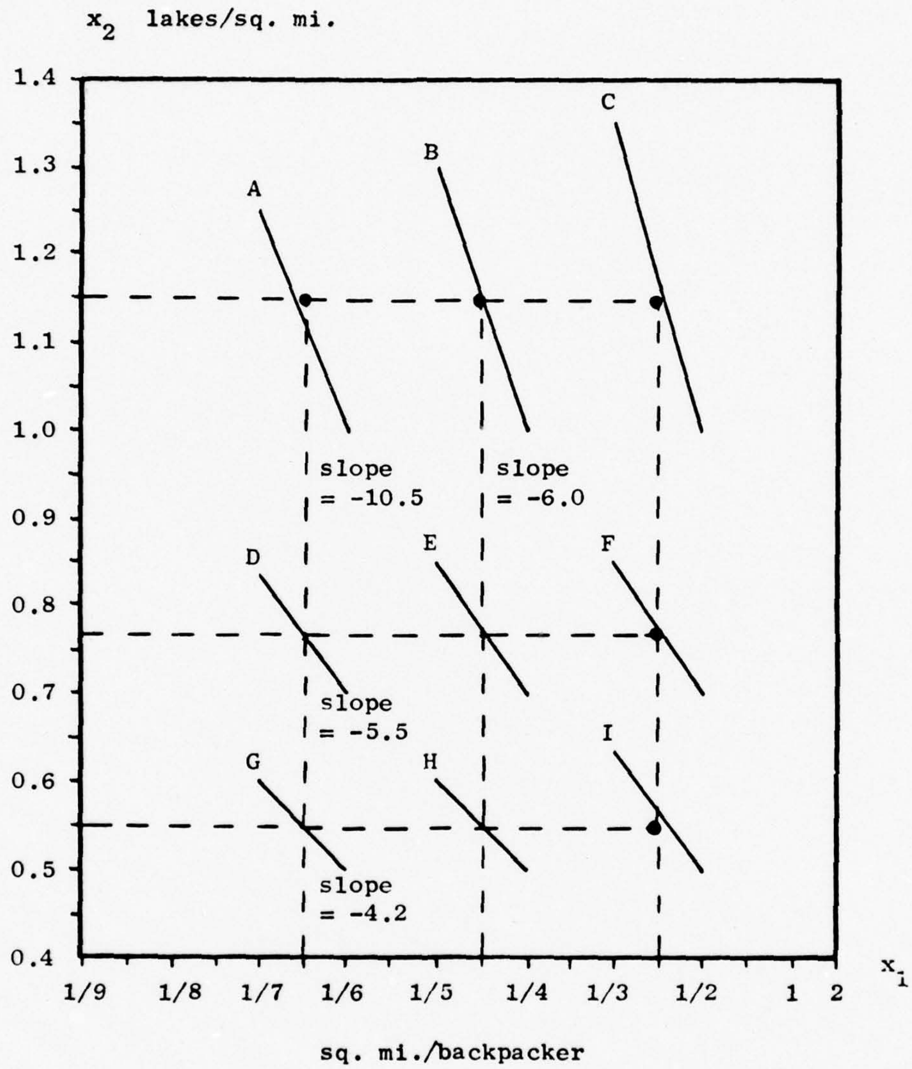

Figure 4.7a. PLOT OF ASSESSMENTS FOR ESTIMATING z_1 AND z_2.

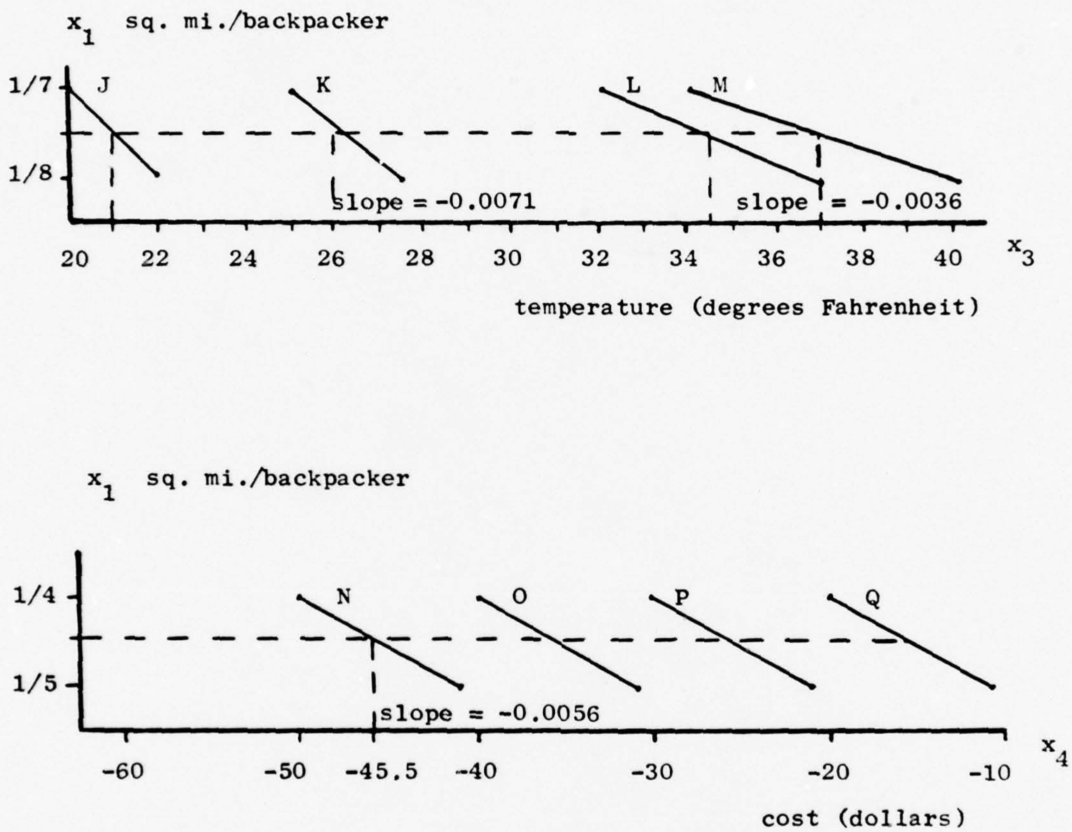

Figure 4.7b. PLOT OF ASSESSMENTS FOR ESTIMATING z_3 AND z_4.

Estimating $z_4(x_4)$ is easy: line segments N, O, P, and Q all have the same slope. This indicates that $\lambda_{41}(\underline{x})$ does not depend on x_4, and so by definition $z_4(x_4) = 0$.

To estimate $z_3(x_3)$ we use line segments J, K, L, and M and the exponential estimating formula from Figure 2.7. For line segments K and L in Figure 4.7b this yields

$$z_3(26) \approx \frac{1}{34.5 - 26} \ln \left(\frac{0.0071}{0.0036}\right) \approx 0.08$$

This estimate appears in Table 4.5. The other estimates in this table can be calculated in the same way from the line segments indicated.

The numbers in this table do not clearly indicate that $z_3(x_3)$ and $z_2(x_2)$ are either increasing or decreasing. We shall use representative constant values for each: $z_3(x_3) = 0.07$ and $z_2(x_2) = 1.7$.

The estimates for $z_1(1/4.5)$ are smaller than the estimates for $z_1(1/6.5)$. So $z_1(x_1)$ is apparently decreasing; we shall use the simple form $z_1(x_1) = (1+\alpha_1)/x_1$, which corresponds to a power form for $v_1(x_1)$. The power form estimates (see Figure 2.7) for z_1 appear in Table 4.6. Using the representative value $z_1(1/4.5) = 8.1$ implies that $z_1(x_1) = 1.8/x_1$.

Applying Theorem 4.1

$$v(\underline{x}) = -a_1 x_1^{-0.8} - a_2 e^{-1.7x_2} - a_3 e^{-0.07x_3} + a_4 x_4$$

To find the scaling constants, we let $a_1 = 1$ and solve for the others using three marginal rates of substitution from Figures 4.7a and 4.7b:

$$\lambda_{12}(1/4.5,\ 1.15,\ x_3,\ x_4) = 6.0$$

Table 4.5

EXPONENTIAL ESTIMATES FOR z_1, z_2, AND z_3

$z_1(1/6.5)$	8.2 (A,B)	8.9 (D,E)	10.9 (G,H)
$z_1(1/4.5)$	5.9 (B,C)	6.8 (E,F)	5.2 (H,I)

$z_2(0.55)$	1.2 (D,G)	1.8 (E,H)	0.5 (F,I)
$z_2(0.77)$	1.7 (A,D)	1.8 (B,E)	2.2 (C,F)

$z_3(21)$	0.04 (J,K)
$z_3(26)$	0.08 (K,L)
$z_3(34)$	0.07 (L,M)

Table 4.6

POWER FORM ESTIMATES FOR z_1

$z_1(1/6.5)$	9.9 (A,B)	10.7 (D,E)	13.1 (G,H)
$z_1(1/4.5)$	8.1 (B,C)	9.2 (E,F)	7.0 (H,I)

$$\lambda_{31}(1/7.5,\ x_2,\ 26,\ x_4) = 0.0071$$

and

$$\lambda_{41}(1/4.5,\ x_2,\ x_3,\ -45.5) = 0.0056$$

This yields

$$v(\underline{x}) = -x_1^{-0.8} - 8\, e^{-1.7x_2} - 19\, e^{-0.07x_3} + 0.066\, x_4 \qquad (4.2)$$

Indifference curves for this value function appear in Figures 4.8a, 4.8b, 4.8c, 4.8d, 4.8e, and 4.8f. Unlabeled solid line segments in these figures correspond to assessments that we did not use specifically in determining (4.2). The indifference curves in Figure 4.8d appear concave. This is an effect of the reciprocal scale used for plotting x_1: it is easily verified that all six substitution aversion coefficients for (4.2) are strictly positive.

Attribute x_3 has constant marginal value preservation tolerance: $\zeta_{3j}(\underline{x}) = 1/0.07 = 14.3$ degrees whenever $j \neq 3$. Thus, according to Figure 2.8, increasing temperature by 14.3 degrees decreases its marginal value in terms of any other attribute by the factor 1/e: this interpretation holds anywhere in Figures 4.8b, 4.8c, and 4.8f. In Figure 4.8f, for example, $\lambda_{34}(\underline{x}) = \$4.63$/degree when $x_3 = 21$ degrees. Therefore, when $x_3 = 21 + 14.3 = 35.3$ degrees, we know that $\lambda_{34}(\underline{x}) = 4.63/e = \$1.70$/degree.

Likewise, attribute x_2 has constant marginal value preservation tolerance: $\zeta_{2j}(\underline{x}) = 1/1.7 = 0.59$ lakes/sq. mi. This can be interpreted in Figures 4.8a, 4.8c, and 4.8e.

Clearly, Equation (4.2) contains a tremendous amount of preference information: far more than we could ever hope to assess.

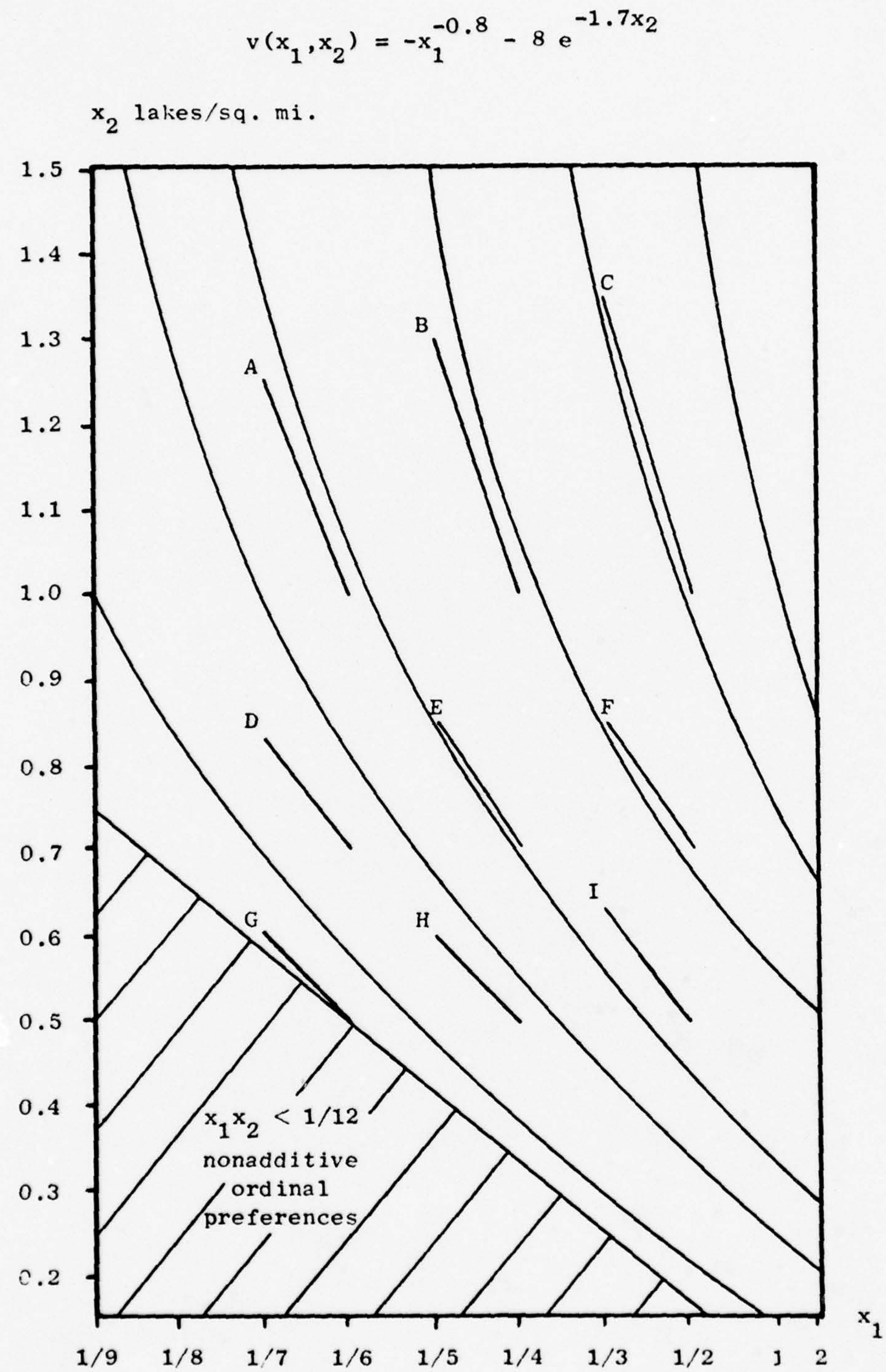

Figure 4.8a. TRADEOFFS BETWEEN BACKPACKERS AND LAKES.

$$v(x_1, x_3) = -x_1^{-0.8} - 19\, e^{-0.07x_3}$$

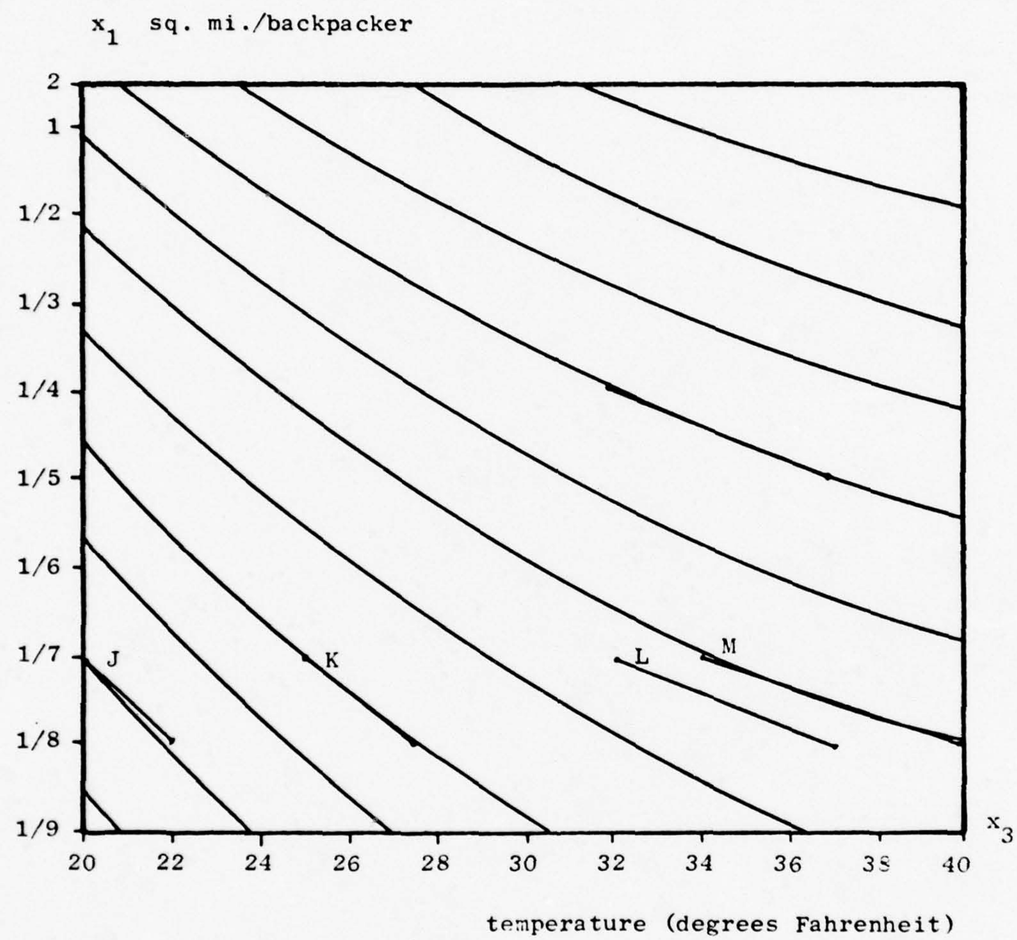

Figure 4.8b. TRADEOFFS BETWEEN BACKPACKERS AND TEMPERATURE.

$$v(x_2,x_3) = -8\,e^{-1.7x_2} - 19\,e^{-0.07x_3}$$

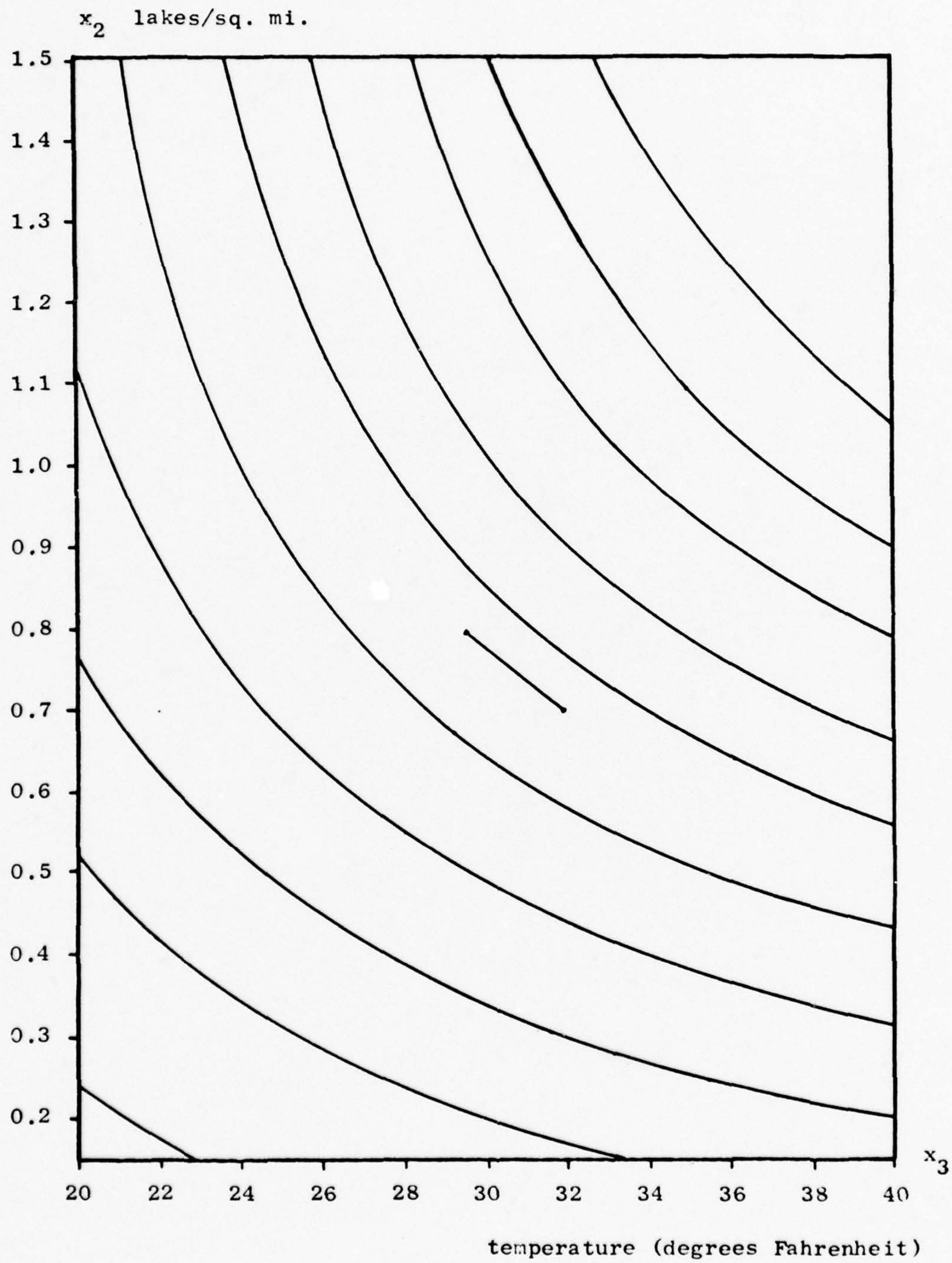

Figure 4.8c. TRADEOFFS BETWEEN LAKES AND TEMPERATURE.

$$v(x_1, x_4) = -x_1^{-0.8} + 0.066\, x_4$$

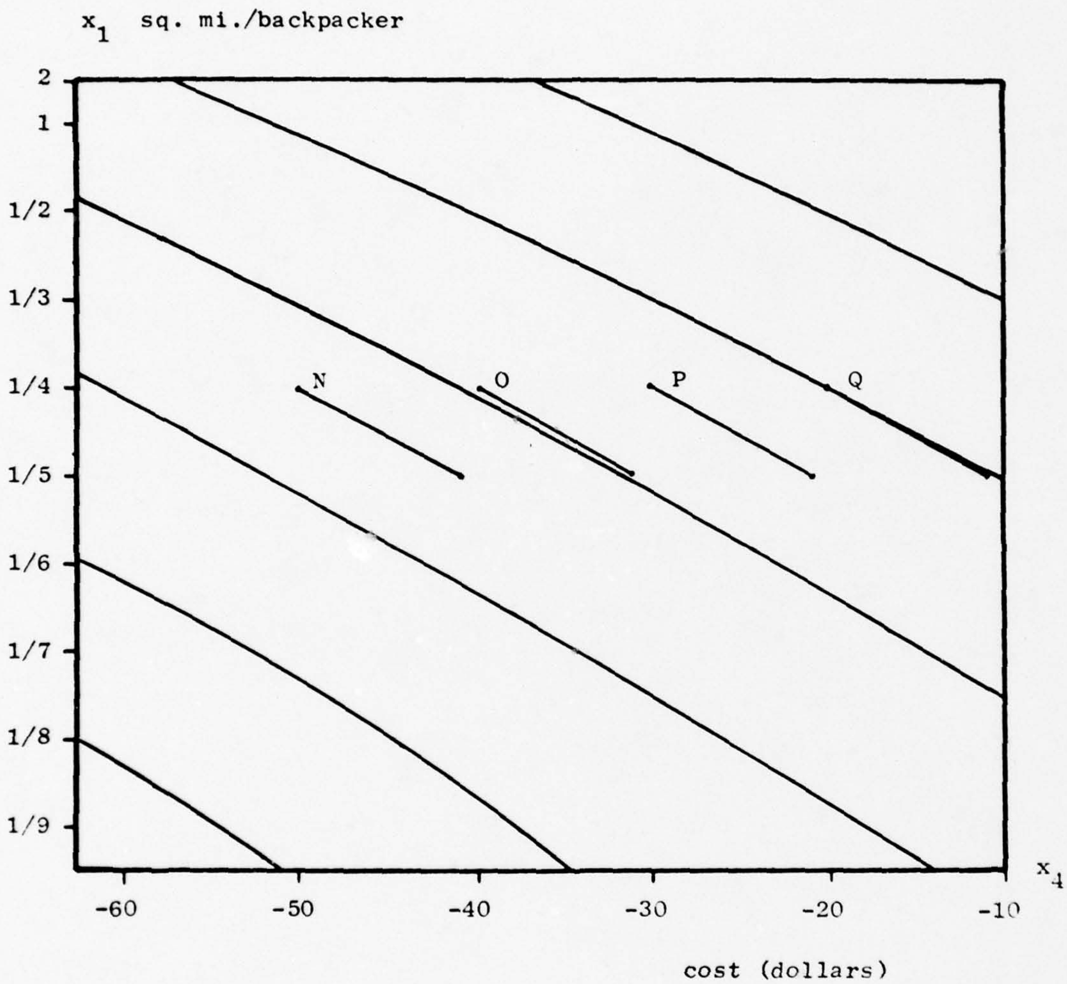

Figure 4.8d. TRADEOFFS BETWEEN BACKPACKERS AND COST.

$$v(x_2,x_4) = -8\,e^{-1.7x_2} + 0.066\,x_4$$

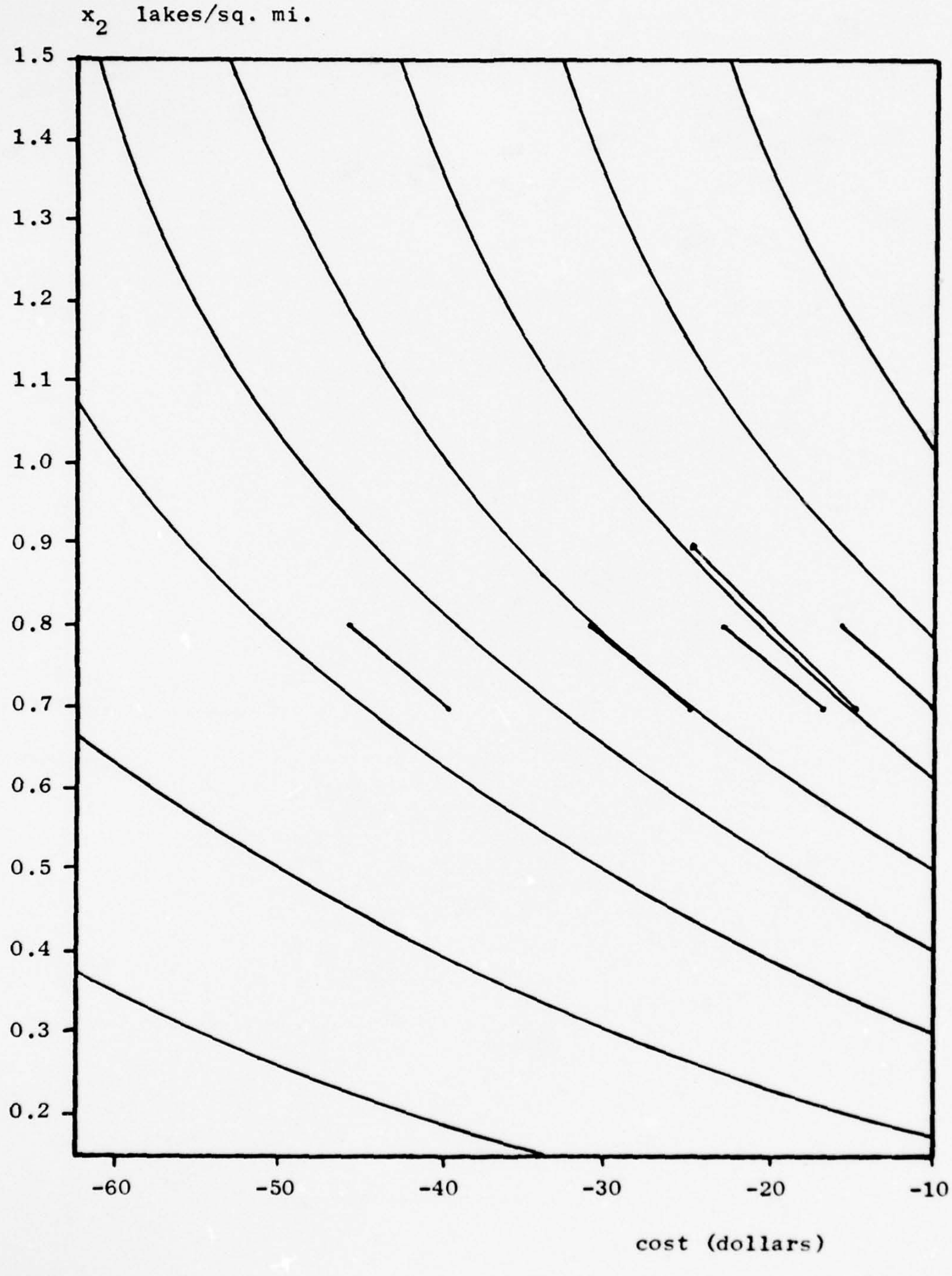

Figure 4.8e. TRADEOFFS BETWEEN LAKES AND COST.

$$v(x_3,x_4) = -19\, e^{-0.07x_3} + 0.066\, x_4$$

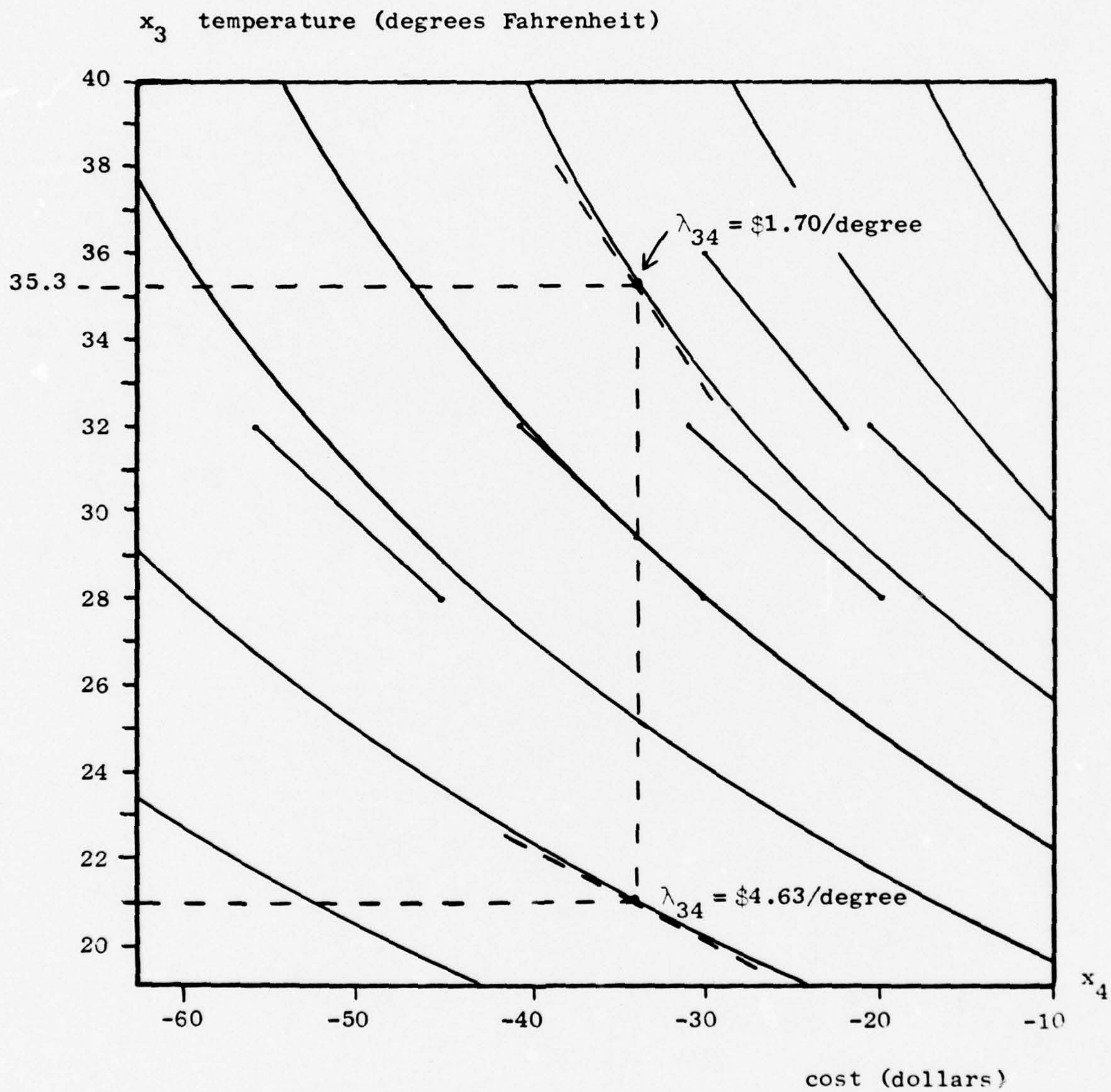

Figure 4.8f. TRADEOFFS BETWEEN TEMPERATURE AND COST.

In Table 4.7 we compare the decision maker's initial response with those implied by $v(\underline{x})$.

To see whether this value function accurately represents the decision maker's preferences, we asked him to suppose that deterministic outcomes were available for each alternative. Then we asked, "Would you be just as happy to let someone else make the decision so long as that person would decide in accordance with (4.2)? If not, how would you modify the indifference curves in Figures 4.8a, 4.8b, 4.8c, 4.8d, 4.8e, or 4.8f?"

After nearly an hour of examining these indifference curves, the decision maker acknowledged that they fairly represent his preferences and that, on that basis, he would be willing to delegate responsibility for a deterministic decision.

Assessing Risk Preference

With ordinal preferences established, assessing risk preference is a relatively simple matter. Combining (4.1) and (4.2), the decision maker's utility function is

$$u(\underline{x}) = -a\ e^{-\gamma(-x_1^{-0.8} - 8\ e^{-1.7x_2} - 19\ e^{-0.07x_3} + 0.066x_4)}$$

To determine γ note that the responses in Figure 4.6 imply

$$-e^{\gamma(1/4.5)^{-0.8}} = -\frac{1}{2}\ e^{\gamma(1/2)^{-0.8}} - \frac{1}{2}\ e^{\gamma(1/6)^{-0.8}}$$

This equation is approximately satisfied by $\gamma = 0.5$. Using this value and letting $a = 1$, the decision maker's multiattribute utility function becomes

Table 4.7

COMPARISON OF $v(\underline{x})$ TO INITIAL ASSESSMENTS

	Line Segment	Indifference Points	Response Implied by $v(\underline{x})$
(x_1, x_2) Indifference Points	A	(1/6, 1.0) ~ (1/7, $\underline{1.25}$)	1.28
	B	(1/4, 1.0) ~ (1/5, $\underline{1.30}$)	1.31
	C	(1/2, 1.0) ~ (1/3, $\underline{1.35}$)	1.36
	D	(1/6, 0.70) ~ (1/7, $\underline{0.83}$)	0.85
	E	(1/4, 0.70) ~ (1/5, $\underline{0.85}$)	0.86
	F	(1/2, 0.70) ~ (1/3, $\underline{0.85}$)	0.89
	G	(1/6, 0.50) ~ (1/7, $\underline{0.60}$)	0.60
	H	(1/4, 0.50) ~ (1/5, $\underline{0.60}$)	0.61
	I	(1/2, 0.50) ~ (1/3, $\underline{0.63}$)	0.63
(x_1, x_3) Indifference Points	J	(1/7, 20) ~ (1/8, $\underline{22}$)	21.7
	K	(1/7, 25) ~ (1/8, $\underline{27.5}$)	27.5
	L	(1/7, 32) ~ (1/8, $\underline{37}$)	36.4
	M	(1/7, 34) ~ (1/8, $\underline{40}$)	39.2
(x_1, x_4) Indifference Points	N	(1/4, -50) ~ (1/5, $\underline{-41}$)	-41.0
	O	(1/4, -40) ~ (1/5, $\underline{-31}$)	-31.0
	P	(1/4, -30) ~ (1/5, $\underline{-21}$)	-21.0
	Q	(1/4, -20) ~ (1/5, $\underline{-11}$)	-11.0

$$u(\underline{x}) = -e^{0.5x_1^{-0.8}+4\,e^{-1.7x_2}+9.5\,e^{-0.07x_3}-0.033x_4} \qquad (4.3)$$

Some of the local preference measures for (4.3) appear in Table 4.8. Note that this utility function is multiattribute risk averse for each pair of attributes and that the first three single-attribute risk aversion coefficients are positive and decreasing. The decision maker's risk aversion on cost is constant: $\rho_4(\underline{x}) = 1/0.033 \approx \$30$.

While this value of risk tolerance is smaller than expected for lotteries over money, it is quite consistent with Steve's previous answers and was found consistent with some simple lottery questions over the range of possible outcomes. Even so, we would expect to find a larger risk tolerance for money if the decision maker were routinely evaluating lotteries with large dollar outcomes.

The other measures of risk aversion were also found consistent using a number of simple lottery questions. On this basis the decision maker agreed that (4.3) is a fair representation of his preferences and that he would be willing to delegate it for use in evaluating alternatives with uncertain outcomes.

Conclusion

In assessing this four attribute utility function, we used a total of 34 preference assessments, excluding consistency checks. Throughout the procedure the decision maker was surprisingly consistent: his answers showed excellent ability to differentiate among the assessment questions. Under these favorable conditions our technology worked very well.

Table 4.8

SOME PREFERENCE MEASURES FOR STEVE HAAS'S UTILITY FUNCTION

Value Function	Single-Attribute Numeraire (x_4)	Utility over Numeraire	Multiattribute Utility Function
$v(\underline{x}) = -x_1^{-0.8}$ $- 8\, e^{-1.7x_2}$ $- 19\, e^{-0.07x_3}$ $+ 0.066x_4$	$\nu(\underline{x}) = 15.2v(\underline{x}) + 98.7$ $\bar{x}_1 = 1/4$ $\bar{x}_2 = 1.0$ $\bar{x}_3 = 32$	$u(\nu) = -e^{-0.033\nu}$	$u(\underline{x}) =$ $-\exp\left[0.5x_1^{-0.8} + 4\, e^{-1.7x_2} + 9.5\, e^{-0.07x_3} - 0.033x_4\right]$
$\lambda_{12}(\underline{x}) = 0.06x_1^{-1.8}\, e^{1.7x_2}$ $\lambda_{13}(\underline{x}) = 0.6x_1^{-1.8}\, e^{0.07x_3}$ $\lambda_{14}(\underline{x}) = 12.1x_1^{-1.8}$		$r_\nu(\underline{x}) = 0.033$	
$z_{1j}(\underline{x}) = \frac{1.8}{x_1}$ $\quad j = 2,3,4$ $z_{2j}(\underline{x}) = 1.7$ $\quad j = 1,3,4$ $z_{4j}(\underline{x}) = 0.0$ $\quad j = 1,2,3,4$		$r_1(\underline{x}) = \frac{1}{x_1}\left[1.8 + 0.4\left(\frac{1}{x_1}\right)^{0.8}\right]$ $r_2(\underline{x}) = 1.7 + 6.8\, e^{-1.7x_2}$ $r_3(\underline{x}) = 0.07 + 0.67\, e^{-0.07x_3}$	
$s_{12}(\underline{x}) = \frac{1}{x_1}\left(1.8 + 0.1x_1^{-0.8}\, e^{1.7x_2}\right)$ $s_{13}(\underline{x}) = \frac{1}{x_1}\left(1.8 + 0.04x_1^{-0.8}\, e^{0.07x_3}\right)$ $s_{14}(\underline{x}) = \frac{1.8}{x_1}$		$r_{12}(\underline{x}) = 83x_1^{-1.8}\, e^{-1.7x_2}$ $r_{13}(\underline{x}) = 8.1x_1^{-1.8}\, e^{-0.07x_3}$ $r_{14}(\underline{x}) = 0.4x_1^{-1.8}$	
Deterministic Additive Independence		Mutual Utility Independence	

By design, this example avoided our main limitation: it is not easy to find a set of attributes that completely describes the possible outcomes for each alternative and that, simultaneously, satisfies deterministic additive independence. Often when the first condition is satisfied the second is not. We outline two research areas for handling this limitation in the next chapter.

Chapter 5

SUGGESTIONS FOR RESEARCH

Existing procedures for modeling and assessing multiattribute preferences are far from complete. In this chapter we suggest two research topics that could lead to important practical contributions.

5.1 Modeling Multiattribute Outcomes

Modeling the possible outcomes is a crucial step in decision analysis. In a good model the outcomes are accurately represented by an attribute or set of attributes over which preferences are easy to assess.

To model each important outcome dimension as an attribute and proceed immediately to preference assessment is usually a mistake. Often, simple relationships among the outcome dimensions can reduce the number of attributes and greatly simplify preference assessment.

This example was suggested by James E. Matheson: consider the entrepreneur's decision of whether to engage in the sale of a new product. He says that he is concerned about three attributes:

x_1 = the number of units that he will sell

x_2 = his revenue per unit

x_3 = his cost per unit

If the entrepreneur is actually concerned only with profit $v(\underline{x}) = x_1(x_2 - x_3)$ while his analyst is naively trying to assess preferences over (x_1, x_2, x_3), the analyst will soon find out that his client's preferences are not additive: for

$$v(\underline{x}) = x_1(x_2 - x_3) \tag{5.1}$$

$$z_{21}(\underline{x}) = \frac{1}{x_2 - x_3} \neq z_2(x_2)$$

Lacking the model (5.1), preference assessment over (x_1, x_2, x_3) would bog down, since we have yet to develop good methods for assessing nonadditive ordinal preferences. With the model (5.1) this difficulty would be avoided, because it is a relatively simple matter to assess preferences over the one-dimensional profit outcome v.

A unified approach to modeling multiattribute outcomes does not yet exist. One method is to construct an economic model as in the entrepreneur's decision above. Another is to arrange the outcome dimensions in a hierarchy to help isolate the important attributes (see Keeney and Raiffa (1975)).

A third approach, which merits investigation, is to use influence diagrams to graph the important preferential dependencies among outcome dimensions. Examining preferential dependencies in this way might suggest where joint preferences can be easily assessed and where economic modeling is desirable.

We believe that modeling multiattribute outcomes is not entirely problem specific and that a synthesis of the three approaches mentioned above could lead to important practical guidelines.

5.2 Nonadditive Value Functions

In many problems the attributes are not related through any simple underlying model. If, at the same time, they fail to satisfy deterministic additive independence, we do not currently have good assessment procedures.

This dissertation provides an anatomy of additive value functions: it consists of marginal value reduction coefficients and scaling constants, all of which we can assess. Likewise, it must be possible to identify anatomies for classes of nonadditive value functions and to extend the multiattribute preference assessment flow chart, which now ends with Figure 4.3c.

A practical problem involving nonadditive ordinal preferences might be a good entrance to this research. Our experience suggests that these problems are common: for the backpacking outcomes in the previous section, how should we assess ordinal preferences when there are more than 12 backpackers/lake (see Figure 4.8a)? What information would be useful? What questions should we ask?

Appendix A

NOTATION

General

x	one-dimensional outcome variable
$\underline{x}$	vector of outcome variables $(x_1,\ldots,x_n)$
$\underline{x}'$	the transpose of $\underline{x}$
$\bar{\underline{x}}$	a fixed level of $\underline{x}$
$\underline{x}_i$	$(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$
$\underline{x}_{ij}$	$(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1},x_{j+1},\ldots,x_n)$
e	the base of natural logarithms (2.71828...)
$>$	"is greater than"
$\succ$	"is strictly preferred to"
$\sim$	"is indifferent to"

Preference and Probability Functions

$v(\underline{x})$	value function (ordinal preference function)
$\nu(\underline{x})$	numeraire function
$u(\underline{x})$	utility function (risk preference function)
$\{\underline{x}\|\&\}$	joint probability distribution over $\underline{x}$ given the state of information $\&$

Preference Measures

$\lambda_{ij}(\underline{x}),\lambda$	marginal rate of substitution of x_j for x_i at $\underline{x}$

$z_{ij}(\underline{x})$	marginal value reduction coefficient
$z_i(x_i)$	$z_{ij}(\underline{x})$ when $v(\underline{x})$ can be written in an additive form
$\zeta_{ij}(\underline{x})$	marginal value preservation tolerance, reciprocal of $z_{ij}(\underline{x})$
$s_{ij}(\underline{x})$	substitution aversion coefficient
$\sigma_{ij}(\underline{x})$	substitution tolerance, reciprocal of $s_{ij}(\underline{x})$
$r(x)$	Pratt's risk aversion coefficient
$\rho(x)$	risk tolerance, reciprocal of $r(x)$
$r_\nu(\underline{x})$	numeraire risk aversion coefficient
$\rho_\nu(\underline{x})$	numeraire risk tolerance, reciprocal of $r_\nu(\underline{x})$
$r_i(\underline{x})$	single-attribute risk aversion coefficient
$\rho_i(\underline{x})$	single-attribute risk tolerance, reciprocal of $r_i(\underline{x})$
$r_{ij}(\underline{x})$	multiattribute risk aversion coefficient
Φ	substitution premium
$\hat{i}$	interest rate
β	discount rate

<u>Differentiation and Integration</u>

$f'(x)$	df/dx
$f''(x)$	d^2f/dx^2
$\int$	general summation operator
$\int f(x)$	$F(x)$ where $F'(x) = f(x)$

Appendix B

PROOF THAT A MULTIVARIATE NORMAL LOTTERY UNDER A PRODUCT-OF-EXPONENTIALS UTILITY FUNCTION SATISFIES THE NUMERAIRE CERTAIN EQUIVALENT APPROXIMATION EXACTLY

For the sum-of-exponentials utility function and any lottery $\{\underline{x}|\mathcal{S}\}$, a closed form diagonal numeraire certain equivalent is given by Equation (3.2):

$$\tilde{v} = -\frac{1}{\sum_{i=1}^{n} \gamma_i} \ln f_{\underline{x}}^{e}(\underline{\gamma})$$

where $f_{\underline{x}}^{e}(\underline{s})$ is the Laplace transform of $\{\underline{x}|\mathcal{S}\}$. The Laplace transform for a multivariate normal lottery with mean $\underline{\bar{x}} = (\bar{x}_1, \ldots, \bar{x}_n)'$ and variance-covariance matrix

$$A = [a_{ij}] \qquad \text{where} \quad a_{ij} = \mathrm{cov}(x_i, x_j)$$

is given by

$$f_{\underline{x}}^{e}(\underline{s}) = e^{-\underline{s}\underline{\bar{x}} + \frac{1}{2}\underline{s}A\underline{s}'} \qquad \text{where} \quad \underline{s} = (s_1, \ldots, s_n)$$

Therefore,

$$\tilde{v} = -\frac{1}{\sum_{i=1}^{n} \gamma_i} \left(-\underline{\gamma}\underline{\bar{x}} + \frac{1}{2} \underline{\gamma}A\underline{\gamma}'\right)$$

$$\tilde{v} = \frac{\sum_{i=1}^{n} \gamma_i \bar{x}_i}{\sum_{i=1}^{n} \gamma_i} - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\gamma_i \gamma_j}{\sum_{k=1}^{n} \gamma_k} a_{ij} \qquad \text{(B.1)}$$

For the product-of-exponentials, the diagonal numeraire is defined by

$$-a\, e^{-\sum_{i=1}^{n} \gamma_i v(\underline{x})} = -a\, e^{-\sum_{i=1}^{n} \gamma_i x_i} \qquad \text{where } a\gamma_i > 0$$

from which

$$v(\underline{x}) = \frac{\sum_{i=1}^{n} \gamma_i x_i}{\sum_{i=1}^{n} \gamma_i} \tag{B.2}$$

$$r_v(\underline{x}) = \sum_{i=1}^{n} \gamma_i$$

Therefore,

$$r_{ij}(\underline{x}) = r_v \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} - \frac{\partial^2 v}{\partial x_i \partial x_j}$$

$$= \frac{\gamma_i \gamma_j}{\sum_{k=1}^{n} \gamma_k} \tag{B.3}$$

Substituting (B.2) and (B.3) into (B.1) yields the desired result:

$$\tilde{v} = v(\bar{\underline{x}}) - \frac{1}{2} \sum_{i,j} r_{ij}(\bar{\underline{x}}) \operatorname{cov}(x_i, x_j)$$

The proof of this result when v is a single-attribute numeraire is analogous. Q.E.D.

Appendix C

DERIVATION OF SOME PREFERENCE MEASURE IDENTITIES

Here we derive Identities (3), (4), (7), (8), and (9) in Figure 2.16. By definition

$$s_{ij}(\underline{x}) = -\frac{\frac{d\lambda}{dx_i}}{\lambda}$$

$$= -\frac{\frac{\partial\lambda}{\partial x_i} + \frac{\partial\lambda}{\partial x_j}\frac{dx_j}{dx_i}}{\lambda}$$

$$= z_{ij}(\underline{x}) + \frac{\partial\lambda}{\partial x_j} \qquad \text{(C.1)}$$

$$\lambda = \frac{\frac{\partial v}{\partial x_i}}{\frac{\partial v}{\partial x_j}}$$

$$\frac{\partial\lambda}{\partial x_j} = \frac{\frac{\partial v}{\partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j} - \frac{\partial v}{\partial x_i}\frac{\partial^2 v}{\partial x_j^2}}{\left(\frac{\partial v}{\partial x_j}\right)^2} \qquad \text{(C.2)}$$

Using Equation (2.1)

$$z_{ji}(\underline{x}) = -\frac{\frac{\partial^2 v}{\partial x_j^2}}{\frac{\partial v}{\partial x_j}} + \frac{\frac{\partial^2 v}{\partial x_i \partial x_j}}{\frac{\partial v}{\partial x_i}}$$

$$\lambda z_{ji}(\underline{x}) = - \frac{\dfrac{\partial v}{\partial x_i}\dfrac{\partial^2 v}{\partial x_j^2}}{\left(\dfrac{\partial v}{\partial x_j}\right)^2} + \frac{\dfrac{\partial^2 v}{\partial x_i \partial x_j}}{\dfrac{\partial v}{\partial x_j}} = \frac{\partial \lambda}{\partial x_j}$$

from Equation (C.2). Substituting this result into (C.1)

$$s_{ij}(\underline{x}) = z_{ij}(\underline{x}) + \lambda z_{ji}(\underline{x}) \tag{C.3}$$

which is Identity (3). Alternatively, using Equation (2.1) we can express $s_{ij}(\underline{x})$ in terms of the value function

$$s_{ij}(\underline{x}) = - \frac{\dfrac{\partial^2 v}{\partial x_i^2}}{\dfrac{\partial v}{\partial x_i}} - \frac{\dfrac{\partial v}{\partial x_i}\dfrac{\partial^2 v}{\partial x_j^2}}{\left(\dfrac{\partial v}{\partial x_j}\right)^2} + 2\,\frac{\dfrac{\partial^2 v}{\partial x_i \partial x_j}}{\dfrac{\partial v}{\partial x_j}} \tag{C.4}$$

Using Equation (C.3)

$$s_{ji}(\underline{x}) = z_{ji}(\underline{x}) + \lambda_{ji}(\underline{x})\, z_{ij}(\underline{x})$$

$$= z_{ji}(\underline{x}) + \frac{1}{\lambda}\, z_{ij}(\underline{x})$$

$$\lambda s_{ji}(\underline{x}) = \lambda z_{ji}(\underline{x}) + z_{ij}(\underline{x}) \tag{C.5}$$

Subtracting (C.5) from (C.3) yields Identity (4):

$$s_{ij}(\underline{x}) = \lambda s_{ji}(\underline{x})$$

To verify Identity (7) consider the utility function $u(v(\underline{x}))$. Since $v(\underline{x})$ expresses an ordinal preference attitude, we can write Equation (2.1)

$$z_{ij}(\underline{x}) = -\frac{\frac{\partial^2 \nu}{\partial x_i^2}}{\frac{\partial \nu}{\partial x_i}} + \frac{\frac{\partial^2 \nu}{\partial x_i \partial x_j}}{\frac{\partial \nu}{\partial x_j}}$$

Subtracting this equation from Equation (2.10) yields

$$r_i(\underline{x}) - z_{ij}(\underline{x}) = r_\nu \frac{\partial \nu}{\partial x_i} - \frac{\frac{\partial^2 \nu}{\partial x_i \partial x_j}}{\frac{\partial \nu}{\partial x_j}} \tag{C.6}$$

Multiplying both sides of this equation by $\partial\nu/\partial x_j$ yields Identity (7):

$$\left[r_i(\underline{x}) - z_{ij}(\underline{x})\right] \frac{\partial \nu}{\partial x_j} = r_\nu \frac{\partial \nu}{\partial x_i} \frac{\partial \nu}{\partial x_j} - \frac{\partial^2 \nu}{\partial x_i \partial x_j} = r_{ij}(\underline{x})$$

Note that $\lambda_{ii}(\underline{x}) = 1$ and, therefore, that $z_{ii}(\underline{x}) = 0$. Thus, rewriting the above equation for $i = j$ yields Identity (8):

$$r_{ii}(\underline{x}) = r_i(\underline{x}) \frac{\partial \nu}{\partial x_i}$$

To verify Identity (9), rewrite (C.6) interchanging i and j

$$r_j(\underline{x}) - z_{ji}(\underline{x}) = r_\nu \frac{\partial \nu}{\partial x_j} - \frac{\frac{\partial^2 \nu}{\partial x_i \partial x_j}}{\frac{\partial \nu}{\partial x_i}}$$

Multiplying both sides by λ

$$\lambda\left[r_j(\underline{x}) - z_{ji}(\underline{x})\right] = r_\nu \frac{\partial \nu}{\partial x_i} - \frac{\frac{\partial^2 \nu}{\partial x_i \partial x_j}}{\frac{\partial \nu}{\partial x_j}} \tag{C.7}$$

Subtracting (C.7) from (C.6) yields Identity (9):

$$r_i(\underline{x}) - z_{ij}(\underline{x}) = \lambda\left[r_j(\underline{x}) - z_{ji}(\underline{x})\right]$$

Appendix D

A CATALOG OF PREFERENCE MEASURES FOR PARTICULAR PREFERENCE FUNCTIONS

Table D.1a

PREFERENCE MEASURES FOR PARTICULAR PREFERENCE FUNCTIONS

Value Function	Single-Attribute Numeraire	Utility over Numeraire	Multiattribute Utility Function
$v(\underline{x}) = \sum_{i=1}^{n} a_i x_i$	$\nu(\underline{x}) = x_1 + \frac{1}{a_1} \sum_{i=2}^{n} a_i (x_i - \bar{x}_i)$	$u(\nu) = -a\, e^{-\gamma\nu}$, $\gamma \neq 0$	$u(\underline{x}) = -a\, e^{-\sum_{i=1}^{n} \gamma_i x_i}$, $\gamma_i = \gamma \frac{a_i}{a_1}$
$\lambda_{ij}(\underline{x}) = \frac{a_i}{a_j}$		$r_\nu(\underline{x}) = \gamma$	
$z_{ij}(\underline{x}) = 0$		$r_i(\underline{x}) = \gamma_i$	
$s_{ij}(\underline{x}) = 0$		$r_{ij}(\underline{x}) = \frac{\gamma_i \gamma_j}{\gamma}$	
Deterministic Additive Independence		Mutual Utility Independence Product-of-Exponentials Delta Property	

Table D.1b

PREFERENCE MEASURES FOR PARTICULAR PREFERENCE FUNCTIONS

Value Function	Single-Attribute Numeraire	Utility over Numeraire	Multiattribute Utility Function
$v(\underline{x}) = \sum_{i=1}^{n} a_i x_i$	$v(\underline{x}) = x_1 + \frac{1}{a_1} \sum_{i=2}^{n} a_i (x_i - \bar{x}_i)$	$u(v) = \begin{cases} -a v^{-\alpha} & \alpha \neq 0 \\ \ln v & \alpha = 0 \end{cases}$	$u(\underline{x}) = \begin{cases} -a[v(\underline{x})]^{-\alpha} & \alpha \neq 0 \\ \ln v(\underline{x}) & \alpha = 0 \end{cases}$
$\lambda_{ij}(\underline{x}) = \frac{a_i}{a_j}$		$r_v(\underline{x}) = \frac{1+\alpha}{v(\underline{x})}$	
$z_{ij}(\underline{x}) = 0$		$r_i(\underline{x}) = \frac{a_i}{a_1} \frac{1+\alpha}{v(\underline{x})}$	
$s_{ij}(\underline{x}) = 0$		$r_{ij}(\underline{x}) = \frac{a_i a_j}{a_1^2} \frac{1+\alpha}{v(\underline{x})}$	
Deterministic Additive Independence			

Table D.1c

PREFERENCE MEASURES FOR PARTICULAR PREFERENCE FUNCTIONS

Value Function	Single-Attribute Numeraire	Utility over Numeraire	Multiattribute Utility Function
$v(\underline{x})$ $= -\sum a_i e^{-\gamma_i x_i}$	$\nu(\underline{x}) = -\frac{1}{\gamma_1} \ln \left[\frac{1}{a_1} \sum a_i e^{-\gamma_i x_i} - \frac{b}{a_1} \right]$ $b = \sum_{i=2}^{n} a_i e^{-\gamma_i \bar{x}_i}$	$u(\nu) = -a e^{-\gamma \nu}$ $\gamma \neq 0$	$u(\underline{x})$ $= -a \left[\frac{1}{a_1} \sum a_i e^{-\gamma_i x_i} - \frac{b}{a_1} \right]^{\frac{\gamma}{\gamma_1}}$
$\lambda_{ij}(\underline{x}) = \frac{a_i \gamma_i}{a_j \gamma_j} e^{-\gamma_i x_i + \gamma_j x_j}$		$r_\nu(\underline{x}) = \gamma$	
$z_{ij}(\underline{x}) = \gamma_i$		$r_i(\underline{x}) = \gamma_i + (\gamma - \gamma_1) \frac{a_i \gamma_i}{a_1 \gamma_1} e^{-\gamma_i x_i + \gamma_1 \nu(\underline{x})}$	
$s_{ij}(\underline{x}) = \gamma_i + \lambda_{ij}(\underline{x}) \gamma_j$		$r_{ij}(\underline{x}) = (\gamma - \gamma_1) \frac{a_i \gamma_i a_j \gamma_j}{(a_1 \gamma_1)^2} e^{-\gamma_i x_i - \gamma_j x_j + 2\gamma_1 \nu(\underline{x})}$	
Deterministic Additive Independence			

Table D.1d

PREFERENCE MEASURES FOR PARTICULAR PREFERENCE FUNCTIONS

Value Function	Single-Attribute Numeraire	Utility over Numeraire	Multiattribute Utility Function
$v(\underline{x}) = -\sum_{i=1}^{n} a_i x_i^{-\alpha_i}$	$\nu(\underline{x}) = \left[\frac{1}{a_1} \sum_{i=1}^{n} a_i x_i^{-\alpha_i} - \frac{b}{a_1}\right]^{-\frac{1}{\alpha_1}}$ $b = \sum_{i=2}^{n} a_i \bar{x}_i^{-\alpha_i}$	$u(\nu) = \begin{cases} -a\nu^{-\alpha} & \alpha \neq 0 \\ \ln \nu & \alpha = 0 \end{cases}$	$u(\underline{x}) = \begin{cases} -a[\nu(\underline{x})]^{-\alpha} & \alpha \neq 0 \\ \ln \nu(\underline{x}) & \alpha = 0 \end{cases}$
$\lambda_{ij}(\underline{x}) = \frac{a_i\alpha_i}{a_j\alpha_j} \frac{x_j^{\alpha_j+1}}{x_i^{\alpha_i+1}}$		$r_\nu(\underline{x}) = \frac{1+\alpha}{\nu(\underline{x})}$	
$z_{ij}(\underline{x}) = \frac{1+\alpha_i}{x_i}$		$r_i(\underline{x}) = \frac{1+\alpha_i}{x_i} + (\alpha - \alpha_1) \frac{a_i\alpha_i}{a_1\alpha_1} x_i^{-\alpha_i-1} [\nu(\underline{x})]^{\alpha_1}$	
$s_{ij}(\underline{x}) = \frac{1+\alpha_i}{x_i} + \lambda_{ij}(\underline{x}) \frac{1+\alpha_j}{x_j}$		$r_{ij}(\underline{x}) = (\alpha - \alpha_1) \frac{a_i\alpha_i a_j\alpha_j}{(a_1\alpha_1)^2} x_i^{-\alpha_i-1} x_j^{-\alpha_j-1} [\nu(\underline{x})]^{2\alpha_1+1}$	
Deterministic Additive Independence			

Table D.1e

PREFERENCE MEASURES FOR PARTICULAR PREFERENCE FUNCTIONS

Value Function	Single-Attribute Numeraire	Utility over Numeraire	Multiattribute Utility Function
$v(\underline{x}) = -\sum_{i=1}^{n} a_i x_i^{-\alpha_i}$	$\nu(\underline{x}) = \left[\frac{1}{a_1} \sum_{i=1}^{n} a_i x_i^{-\alpha_i} - \frac{b}{a_1}\right]^{-\frac{1}{\alpha_1}}$ $b = \sum_{i=2}^{n} a_i \bar{x}_i^{-\alpha_i}$	$u(\nu) = -a\ e^{-\gamma\nu}$ $\gamma \neq 0$	$u(\underline{x}) = -a\ e^{-\gamma\nu(\underline{x})}$
$\lambda_{ij}(\underline{x}) = \frac{a_i\alpha_i}{a_j\alpha_j} \frac{x_j^{\alpha_j+1}}{x_i^{\alpha_i+1}}$		$r_\nu(x) = \gamma$	
$z_{ij}(\underline{x}) = \frac{1+\alpha_i}{x_i}$		$r_i(\underline{x}) = \frac{1+\alpha_i}{x_i} + \frac{a_i\alpha_i}{a_1\alpha_1} x_i^{-\alpha_i-1}$ $\times [1 + \alpha_1 + \gamma\nu(\underline{x})][\nu(\underline{x})]^{\alpha_1}$	
$s_{ij}(\underline{x}) = \frac{1+\alpha_i}{x_i} + \lambda_{ij}(\underline{x}) \frac{1+\alpha_j}{x_j}$		$r_{ij}(\underline{x}) = \frac{a_i\alpha_i a_j\alpha_j}{(a_1\alpha_1)^2} x_i^{-\alpha_i-1} x_j^{-\alpha_j-1}$ $\times [1 + \alpha_1 + \gamma\nu(\underline{x})][\nu(\underline{x})]^{2\alpha_1+1}$	
Deterministic Additive Independence			

Table D.1f

PREFERENCE MEASURES FOR PARTICULAR PREFERENCE FUNCTIONS

Value Function	Single-Attribute Numeraire	Utility over Numeraire	Multiattribute Utility Function
$v(\underline{x}) = \sum_{i=1}^{n} a_i \ln x_i$	$\nu(\underline{x}) = x_1\left(\frac{x_2}{\bar{x}_2}\right)^{\eta_2} \cdots \left(\frac{x_n}{\bar{x}_n}\right)^{\eta_n}$ $\eta_i = \frac{a_i}{a_1}$	$u(\nu) = -a\, e^{-\gamma\nu}$ $\gamma \neq 0$	$u(x) = -a\, e^{-\gamma x_1\left(\frac{x_2}{\bar{x}_2}\right)^{\eta_2} \cdots \left(\frac{x_n}{\bar{x}_n}\right)^{\eta_n}}$
$\lambda_{ij}(\underline{x}) = \frac{a_i}{a_j} \frac{x_j}{x_i}$		$r_\nu(\underline{x}) = \gamma$	
$z_{ij}(x) = \frac{1}{x_i}$		$r_i(\underline{x}) = \frac{1}{x_i} + \frac{\eta_i}{x_i}[\gamma\nu(\underline{x}) - 1]$	
$s_{ij}(x) = \frac{1}{x_i}\left(1 + \frac{a_i}{a_j}\right)$		$r_{ij}(\underline{x}) = \frac{\eta_i}{x_i}\frac{\eta_j}{x_j}[\gamma\nu(\underline{x}) - 1]\,\nu(\underline{x})$	
Deterministic Additive Independence			

Table D.1g

PREFERENCE MEASURES FOR PARTICULAR PREFERENCE FUNCTIONS

Value Function	Single-Attribute Numeraire	Utility over Numeraire	Multiattribute Utility Function
$v(\underline{x}) = \sum_{i=1}^{n} v_i(x_i)$	$\nu(\underline{x}) = \sum_{i=1}^{n} v_i(x_i)$	$u(\nu) = -a\, e^{-\gamma\nu}$	$u(\underline{x}) = -a\, e^{-\gamma(v_1(x_1)+\dots+v_n(x_n))}$
$\lambda_{ij}(\underline{x}) = \dfrac{\frac{dv_i}{dx_i}}{\frac{dv_j}{dx_j}}$		$r_\nu(\underline{x}) = \gamma$	
$z_{ij}(\underline{x}) = -\dfrac{\frac{d^2v_i}{dx_i^2}}{\frac{dv_i}{dx_i}}$		$r_i(\underline{x}) = z_{ij}(\underline{x}) + \gamma \dfrac{dv_i}{dx_i}$	
$s_{ij}(\underline{x}) = -\dfrac{\frac{d^2v_i}{dx_i^2}}{\frac{dv_i}{dx_i}} - \lambda_{ij}(\underline{x}) \dfrac{\frac{d^2v_j}{dx_j^2}}{\frac{dv_j}{dx_j}}$		$r_{ij}(\underline{x}) = \gamma \dfrac{dv_i}{dx_i} \dfrac{dv_j}{dx_j}$	
Deterministic Additive Independence		Mutual Utility Independence	

Table D.1h

PREFERENCE MEASURES FOR PARTICULAR PREFERENCE FUNCTIONS

Value Function	Single-Attribute Numeraire	Utility over Numeraire	Multiattribute Utility Function
$v(\underline{x}) = -\sum a_i e^{-\gamma_i x_i}$	$\nu(\underline{x}) = -\frac{1}{\gamma_1} ln \left[\frac{1}{a_1}\sum a_i e^{-\gamma_i x_i} - \frac{b}{a_1}\right]$ $b = \sum_{i=2}^{n} a_i e^{-\gamma_i \bar{x}_i}$	$u(\nu) = -a e^{a_1 e^{-\gamma_1 \nu}}$	$u(\underline{x}) = -a e^{\Sigma a_i e^{-\gamma_i x_i}}$
$\lambda_{ij}(\underline{x}) = \frac{a_i \gamma_i}{a_j \gamma_j} e^{-\gamma_i x_i + \gamma_j x_j}$		$r_\nu(\underline{x}) = \gamma_1\left(1 + a_1 e^{-\gamma_1 \nu(\underline{x})}\right)$	
$z_{ij}(\underline{x}) = \gamma_i$		$r_i(\underline{x}) = \gamma_i\left(1 + a_i e^{-\gamma_i x_i}\right)$	
$s_{ij}(\underline{x}) = \gamma_i + \lambda_{ij}(\underline{x})\, \gamma_j$		$r_{ij}(\underline{x}) = \frac{a_i \gamma_i a_j \gamma_j}{a_1 \gamma_1} e^{-\gamma_i x_i - \gamma_j x_j + \gamma_1 \nu(\underline{x})}$	
Deterministic Additive Independence		Mutual Utility Independence	

Table D.1i

PREFERENCE MEASURES FOR PARTICULAR PREFERENCE FUNCTIONS

Value Function	Single-Attribute Numeraire	Utility over Numeraire	Multiattribute Utility Function
$v(\underline{x})$ $= -\sum a_i x_i^{-\alpha_i}$	$\nu(\underline{x}) = \left[\frac{1}{a_1}\sum a_i x_i^{-\alpha_i} - \frac{b}{a_1}\right]^{-\frac{1}{\alpha_1}}$ $b = \sum a_i \bar{x}_i^{-\alpha_i}$ $\alpha_i \neq 0$	$u(\nu) = -a\, e^{a_1 \nu^{-\alpha_1}}$	$u(\underline{x}) = -a\, e^{\Sigma a_i x_i^{-\alpha_i}}$
$\lambda_{ij}(\underline{x}) = \frac{a_i \alpha_i}{a_j \alpha_j} \frac{x_j^{\alpha_j+1}}{x_i^{\alpha_i+1}}$		$r_\nu(\underline{x}) = \frac{1+\alpha_1}{\nu} + a_1\alpha_1[\nu(\underline{x})]^{-\alpha_1-1}$	
$z_{ij}(\underline{x}) = \frac{1+\alpha_i}{x_i}$		$r_i(\underline{x}) = \frac{1+\alpha_i}{x_i} + a_i\alpha_i x_i^{-\alpha_i-1}$	
$s_{ij}(\underline{x}) = \frac{1+\alpha_i}{x_i} + \lambda_{ij}(\underline{x}) \frac{1+\alpha_j}{x_j}$		$r_{ij}(\underline{x}) = \frac{a_i\alpha_i a_j\alpha_j}{a_1\alpha_1} x_i^{-\alpha_i-1} x_j^{-\alpha_j-1} [\nu(\underline{x})]^{1+\alpha_1}$	
Deterministic Additive Independence		Mutual Utility Independence	

Appendix E

PROOF OF DELTA PROPERTIES FOR ADDITIVE UTILITY FUNCTIONS

The method used in this proof is due to Howard (1974). We begin by proving the <u>if</u> part of part (i) of Theorem 3.4. Let $\underline{x}_i = \underline{\bar{x}}_i$. By assumption

$$u_i(\tilde{x}_i + \Delta) = \int_{x_i} \{x_i | \&\} \, u_i(x_i + \Delta)$$

Differentiating with respect to Δ

$$u_i'(\tilde{x}_i + \Delta) = \int_{x_i} \{x_i | \&\} \, u_i'(x_i + \Delta) \tag{E.1}$$

$$u_i''(\tilde{x}_i + \Delta) = \int_{x_i} \{x_i | \&\} \, u_i''(x_i + \Delta) \tag{E.2}$$

Dividing (E.2) by (E.1) and setting $\Delta = 0$,

$$\frac{u_i''(\tilde{x}_i)}{u_i'(\tilde{x}_i)} = \frac{\int_{x_i} \{x_i | \&\} \, u_i''(x_i)}{\int_{x_i} \{x_i | \&\} \, u_i'(x_i)} \tag{E.3}$$

In general, many different lotteries would have the same certain equivalent $\tilde{x}_i$. For all such lotteries the left-hand side of (E.3) is constant. Thus, the derivatives on the right-hand side must be proportional. Defining the proportionality constant $-\gamma_i$,

$$\frac{u_i''(x_i)}{u_i'(x_i)} = -\gamma_i$$

Solving this equation, $u_i(x_i)$ can be written as any positive linear transformation of

$$u_i(x_i) = \begin{cases} -a_i \; e^{-\gamma_i x_i} & \text{where } \gamma_i \neq 0 \\ a_i x_i & \text{where } \gamma_i = 0 \end{cases}$$

To prove the <u>if</u> part of part (ii) of Theorem 3.4, assume that $x_i > 0$ and $\Delta > 0$. Write the lottery condition

$$u_i(\Delta\tilde{x}_i) = \int_{x_i} \{x_i \mid \mathcal{S}\} \; u_i(\Delta x_i)$$

Using an argument analogous to that for part (i), we define the proportionality constant $-(1 + \alpha_i)$ and obtain

$$\frac{u_i''(x_i)}{u_i'(x_i)} = - \frac{1 + \alpha_i}{x_i}$$

Solving this equation, $u_i(x_i)$ can be written as any positive linear transformation of

$$u_i(x_i) = \begin{cases} -a_i x_i^{-\alpha_i} & \text{where } \alpha_i \neq 0 \\ a_i \ln x_i & \text{where } \alpha_i = 0 \end{cases}$$

To show the <u>only if</u> part for the exponential form, observe that if

$$u_i(x_i) = -a_i \; e^{-\gamma_i x_i}$$

then

$$u_i(\tilde{x}_i) = \int_{x_i} \{x_i|\&\}\, u_i(x_i)$$

implies

$$e^{-\gamma_i \triangle} u_i(\tilde{x}_i) = \int_{x_i} \{x_i|\&\}\, e^{-\gamma_i \triangle} u_i(x_i)$$

$$u_i(\tilde{x}_i + \triangle) = \int_{x_i} \{x_i|\&\}\, u_i(x_i + \triangle)$$

In an analogous way, it is easy to verify that the other forms of $u_i(x_i)$ in Theorem 3.4 satisfy their respective lottery conditions.

Q.E.D.

Appendix F

PROOF OF DELTA PROPERTIES FOR MULTIPLICATIVE UTILITY FUNCTIONS

Proof of Theorem 3.9. Product-of-exponentials delta property. Assume

$$u(\tilde{\underline{x}}) = \int_{\underline{x}} \{\underline{x}|\mathcal{S}\}\, u(\underline{x})$$

implies

$$u(\tilde{\underline{x}} + \underline{\Delta}) = \int_{\underline{x}} \{\underline{x}|\mathcal{S}\}\, u(\underline{x} + \underline{\Delta}) \qquad \text{for any } \underline{\Delta} \tag{F.1}$$

Let y be a subset of $x_1,\ldots,x_n$, and let $\Delta_i = 0$ for each attribute contained in y. Assume that attributes not contained in y are known deterministically. Since (F.1) must hold when we vary Δ_i for each attribute not contained in y, it is clear that y is utility independent of its complement. Since this holds for any subset y, attributes $x_1,\ldots,x_n$ are mutually utility independent. Then, by Theorem 3.5, $u(\underline{x})$ must be either additive or multiplicative. By assumption, the decision maker is multiattribute risk averse, and, therefore, by Theorem 3.2, $u(\underline{x})$ cannot be additive. Thus,

$$u(\underline{x}) = u_1(x_1)\, u_2(x_2) \ldots u_n(x_n)$$

Assuming that x_j is known deterministically for $j \neq i$, (F.1) becomes

$$u_i(\tilde{x}_i + \Delta_i) = \int_{x_i} \{x_i|\mathcal{S}\}\, u_i(x_i + \Delta_i)$$

This equation must hold for all $\{x_i|\mathcal{S}\}$ and Δ_i. Thus, as we have shown in Appendix E,

$$u_i(x_i) = \begin{cases} a_i\, e^{-\gamma_i x_i} + b_i & \text{for } \gamma_i \neq 0 \\ a_i x_i + b_i & \text{for } \gamma_i = 0 \end{cases}$$

Risk aversion implies that $\gamma_i > 0$. Thus, $u(\underline{x})$ can be written

$$u(\underline{x}) = \prod_{i=1}^{n} \left[a_i\, e^{-\gamma_i x_i} + b_i\right] \qquad \text{where } \gamma_i > 0 \qquad \text{(F.2)}$$

In order to satisfy (F.1), if it is true for $\triangle_i = \triangle_j = 0$ that

$$\left[a_i\, e^{-\gamma_i(\tilde{x}_i + \triangle_i)} + b_i\right]\left[a_j\, e^{-\gamma_j(\tilde{x}_j + \triangle_j)} + b_j\right]$$

$$= \int_{x_i, x_j} \{x_i, x_j \mid \mathcal{S}\}\left[a_i\, e^{-\gamma_i(x_i + \triangle_i)} + b_i\right]\left[a_j\, e^{-\gamma_j(x_j + \triangle_j)} + b_j\right]$$

then this expression must hold for any $\triangle_i$ and $\triangle_j$. If either $b_i \neq 0$ or $b_j \neq 0$, one can easily construct a case in which this requirement is not satisfied. Thus, $b_i = 0$ for all i, and (F.2) can be rewritten

$$u(\underline{x}) = -a\, e^{-\gamma_1 x_1 - \cdots - \gamma_n x_n} \qquad \text{where } \gamma_i > 0$$

For $\partial u/\partial x_i > 0$, we must have $a > 0$. Then, through a positive linear transformation,

$$u(\underline{x}) = -e^{-\gamma_1 x_1 - \cdots - \gamma_n x_n} \qquad \text{where } \gamma_i > 0$$

To show the <u>only if</u> part, observe that if $u(\underline{x})$ is a product-of-exponentials

$$u(\tilde{\underline{x}}) = \int_{\underline{x}} \{\underline{x}|\mathcal{S}\}\ u(\underline{x})$$

$$u(\tilde{\underline{x}})\ e^{-\gamma\underline{\Delta}} = \int_{\underline{x}} \{\underline{x}|\mathcal{S}\}\ u(\underline{x})\ e^{-\gamma\underline{\Delta}}$$

$$u(\tilde{\underline{x}} + \underline{\Delta}) = \int_{\underline{x}} \{\underline{x}|\mathcal{S}\}\ u(\underline{x} + \underline{\Delta})$$

Q.E.D.

<u>Proof of Theorem 3.8</u>. Product-of-powers delta property. We shall first prove the <u>if</u> part. Using an argument analogous to the one in the proof of the product-of-exponentials delta property above, one can easily verify that $u(\underline{x})$ is multiplicative

$$u(\underline{x}) = u_1(x_1)\ u_2(x_2)\ \ldots\ u_n(x_n)$$

and that for all $\{x_i|\mathcal{S}\}$ and $\Delta_i > 0$

$$u_i(\Delta_i\tilde{x}_i) = \int_{x_i} \{x_i|\mathcal{S}\}\ u_i(\Delta_i x_i)$$

Thus, from Appendix E,

$$u_i(x_i) = \begin{cases} a_i x_i^{-\alpha_i} + b_i & \text{for } \alpha_i \neq 0 \\ a_i \ln x_i + b_i & \text{for } \alpha_i = 0 \end{cases}$$

So, $u(\underline{x})$ can be written

$$u(\underline{x}) = \prod_{\substack{i=1 \\ \alpha_i \neq 0}}^{n} \left[a_i x_i^{-\alpha_i} + b_i\right] \prod_{\substack{j=1 \\ \alpha_j = 0}}^{n} \left[a_j \ln x_j + b_j\right]$$

$$= -a \prod_{\substack{i=1 \\ \alpha_i \neq 0}}^{n} \left[x_i^{-\alpha_i} + c_i\right] \prod_{\substack{j=1 \\ \alpha_j = 0}}^{n} \left[\ln x_j + c_j\right]$$

When $\alpha_j = 0$, changing x_j *from* $e^{-c_j-\epsilon}$ to $e^{-c_j+\epsilon}$ for some small ϵ changes the sign of $u(\underline{x})$. Clearly, $\partial u/\partial x_i$ for $i \neq j$ cannot be positive in both cases. Since we require that $\partial u/\partial x_i$ be positive for all $x_i > 0$, we see that $u(\underline{x})$ must not contain any logarithmic factors. Thus, $\alpha_j \neq 0$, and $u(\underline{x})$ can be written

$$u(\underline{x}) = -a \prod_{i=1}^{n} \left[x_i^{-\alpha_i} + c_i\right]$$

To satisfy the lottery condition in Theorem 3.8, if it is true for $\Delta_i = \Delta_j = 1$ that

$$\left[(\Delta_i \tilde{x}_i)^{-\alpha_i} + c_i\right]\left[(\Delta_j \tilde{x}_j)^{-\alpha_j} + c_j\right]$$

$$= \int_{x_i, x_j} \{x_i, x_j | \mathcal{S}\} \left[(\Delta_i x_i)^{-\alpha_i} + c_i\right]\left[(\Delta_j x_j)^{-\alpha_j} + c_j\right]$$

then this expression must hold for any $\Delta_i > 0$ and $\Delta_j > 0$. If either $c_i \neq 0$ or $c_j \neq 0$, it would be easy to construct a case in which this requirement is not satisfied. Thus, $c_i = 0$ for all i, and $u(\underline{x})$ can be written

$$u(\underline{x}) = -a x_1^{-\alpha_1} x_2^{-\alpha_2} \ldots x_n^{-\alpha_n}$$

For $\partial u/\partial x_i$ to be positive, we require that $a\alpha_i > 0$. For single-attribute risk aversion, we must have $\alpha_i > -1$.

To show the <u>only if</u> part, observe that if $u(\underline{x})$ is a product-of-powers, then

$$u(\underline{\tilde{x}}) = \int_{\underline{x}} \{\underline{x}|\mathscr{S}\}\ u(\underline{x})$$

$$\Delta_1^{-\alpha_1} \cdots \Delta_n^{-\alpha_n} u(\underline{\tilde{x}}) = \int_{\underline{x}} \{\underline{x}|\mathscr{S}\}\ \Delta^{-\alpha_1} \cdots \Delta^{-\alpha_n}\ u(\underline{x})$$

$$u(\Delta_1\tilde{x}_1, \ldots, \Delta_n\tilde{x}_n) = \int_{\underline{x}} \{\underline{x}|\mathscr{S}\}\ u(\Delta_1 x_1, \ldots, \Delta_n x_n)$$

Q.E.D.

Appendix G

PROOF THAT $z_{ij}(\underline{x}) = z_i(x_i)$ IMPLIES AN ADDITIVE VALUE FUNCTION

Suppose that

$$z_{ij}(\underline{x}) = z_i(x_i) \qquad \text{for each } i \text{ and } j \neq i \tag{G.1}$$

$$-\frac{\frac{\partial \lambda}{\partial x_i}}{\lambda} = z_i(x_i)$$

$$\frac{\partial}{\partial x_i} \ln \lambda = -z_i(x_i)$$

$$\ln \lambda = -\int^{x_i} z_i(w)\, dw + g_i(\underline{x}_{\bar{i}})$$

$$\lambda = \frac{e^{-\int^{x_i} z_i(w)\, dw}}{e^{-g(\underline{x}_{\bar{i}})}}$$

Let

$$v_i(x_i) = \int^{x_i} e^{-\int^{y} z_i(w)\, dw}\, dy$$

$$\hat{v}_i(\underline{x}_{\bar{i}}) = \int^{x_j} e^{-g(y,\, \underline{x}_{\bar{i}j})}\, dy$$

Therefore, since

$$\lambda = \frac{\frac{\partial v}{\partial x_i}}{\frac{\partial v}{\partial x_j}}$$

we can see that

$$v(\underline{x}) = v_i(x_i) + \hat{v}_i(\underline{x}_{\bar{i}})$$

is consistent with the decision maker's preferences. For $n = 2$, this proves that $v(\underline{x})$ can be written in an additive form. Using induction, assume that $v(\underline{x})$ is additive for $n-1$ attributes. Then, for n attributes we can write

$$v(x_1, \ldots, x_n) = \bar{v}_1(x_1, x_n) + \ldots + \bar{v}_{n-1}(x_{n-1}, x_n) \tag{G.2}$$

as can be seen by fixing x_n and considering $v(x_1, \ldots, x_{n-1}, x_n)$ to be an additive value function over $(x_1, \ldots, x_{n-1})$. Then, using Equations (2.1) and (G.1)

$$z_{in}(\underline{x}) = - \frac{\dfrac{\partial^2 \bar{v}_i}{\partial x_i^2}}{\dfrac{\partial \bar{v}_i}{\partial x_i}} + \frac{\dfrac{\partial^2 \bar{v}_i}{\partial x_i \partial x_n}}{\sum_{j=1}^{n-1} \dfrac{\partial \bar{v}_j}{\partial x_n}} = z_i(x_i) \qquad \text{for } i = 1, \ldots, n-1$$

$$z_{ij}(\underline{x}) = - \frac{\dfrac{\partial^2 \bar{v}_i}{\partial x_i^2}}{\dfrac{\partial \bar{v}_i}{\partial x_i}} = z_i(x_i) \qquad \text{for } j \neq i, n$$

Subtracting and simplifying,

$$\frac{\partial^2 \bar{v}_i}{\partial x_i \partial x_n} = 0$$

$$\frac{\partial \bar{v}_i}{\partial x_n} = g_i'(x_n)$$

$$\bar{v}_i(x_i, x_n) = g_i(x_n) + v_i(x_i) \tag{G.3}$$

Define

$$v_n(x_n) = \sum_{i=1}^{n-1} g_i(x_n) \tag{G.4}$$

Substituting (G.3) and (G.4) into (G.2), shows that the induction holds:

$$v(\underline{x}) = v_1(x_1) + \ldots + v_n(x_n)$$

Q.E.D.

Appendix H

DERIVATION OF A POWER FORM ESTIMATE FOR THE MARGINAL VALUE REDUCTION COEFFICIENT

Assume that

$$v(\underline{x}) = v_i(x_i) + w_i(\underline{x}_{-i})$$

and that $v_i(x_i)$ has the power form

$$v_i(x_i) = \begin{cases} -a_i x_i^{-\alpha_i} & \alpha_i \neq 0,\ x_i > 0 \\ a_i \ln x_i & \alpha_i = 0,\ x_i > 0 \end{cases}$$

Using Equation (2.1)

$$z_{ij}(\underline{x}) = \frac{1 + \alpha_i}{x_i} \qquad \text{for} \quad j \neq i$$

From Equation (2.2)

$$\frac{\bar{\lambda}}{\hat{\lambda}} = \frac{v_i'(\bar{x}_i)}{v_i'(\bar{x}_i + \Delta)}$$

$$= \frac{\bar{x}_i^{-\alpha_i - 1}}{(\bar{x}_i + \Delta)^{-\alpha_i - 1}} = \left(1 + \frac{\Delta}{\bar{x}_i}\right)^{1+\alpha_i}$$

$$\ln \frac{\bar{\lambda}}{\hat{\lambda}} = (1 + \alpha_i) \ln \left(1 + \frac{\Delta}{\bar{x}_i}\right)$$

$$\frac{1 + \alpha_i}{\bar{x}_i} = \frac{\ln \frac{\bar{\lambda}}{\hat{\lambda}}}{\bar{x}_i \ln \left(1 + \frac{\Delta}{\bar{x}_i}\right)}$$

Thus,

$$z_{ij}(\underline{\bar{x}}) = \frac{\ln \dfrac{\bar{\lambda}}{\hat{\lambda}}}{\bar{x}_i \ln\left(1 + \dfrac{\Delta}{\bar{x}_i}\right)}$$

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An important decision may hinge on the decision maker's preferences over the possible outcomes. In many problems, the possible outcomes will appear to have several important dimensions. This dissertation provides some fundamental tools and methods for interpreting and assessing preferences

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over multiattribute outcomes. There are three steps in the development.

First, six preference measures are presented as tools for understanding and analyzing multiattribute preference functions. Each preference measure has a simple interpretation, describes a fundamental aspect of local preference attitude, and is well-defined for any smooth preference function.

Second, a hierarchy of independence conditions and delta properties is identified to provide an easy method for limiting the functional expression of a preference attitude. Each independence condition and delta property is easy to interpret and apply during assessment. The weakest independence condition that we consider, deterministic additive independence, requires that an additive form exist for ordinal preferences.

The six preference measures along with the independence conditions and delta properties stand together as a powerful technology for interpreting any smooth multiattribute preference function.

In the third step, the preference measures, independence conditions, and delta properties are assembled into a practical assessment procedure for multiattribute preference functions. This procedure is concisely summarized in a flow chart; it applies for any preference attitude that satisfies deterministic additive independence. A multiattribute risk preference function is assessed for an actual decision maker as an illustration.